This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

**Usage guidelines**

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

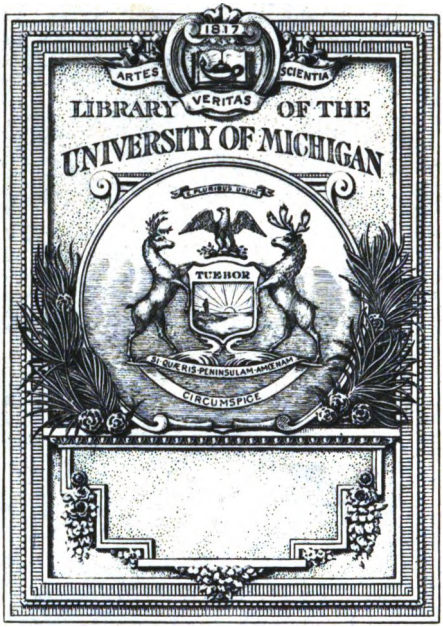
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

**About Google Book Search**

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`




Digitized by Google

QA
35
.S59
1792

Digitized by Google

Digitized by Google

A. ~ S. Webber.

# SELECT EXERCISES FOR YOUNG PROFICIENTS IN THE MATHEMATICKS.

CONTAINING

I. A great variety of Algebraical Problems with their Solutions.

II. A choice Number of Geometrical Problems with their Solutions both Algebraical and Geometrical.

III. The Theory of Gunnery, independent of the Conic Sections.

IV. A new and very comprehenſive Method for finding the Roots of Equations in Numbers.

V. A ſhort Account of the Nature and firſt Principles of Fluxions.

By THOMAS SIMPSON, F. R. S.

A NEW EDITION.

TO WHICH IS PREFIXED,

An Account of the Life and Writings of the Author,

By CHARLES HUTTON, F. R. S.

LONDON:

Printed for F. WINGRAVE,
Succeſſor to Mr. NOURSE, in the *Strand*.
MDCCXCII.

Digitized by Google

Digitized by Google

Hist. of sci.
Dudley
5-6-46
5437 3

TO

# JOHN BACON,

Of *Newtoncap*, Esq; F. R. S.

SIR,

WHEN Gentlemen of your Station and Figure become the Patrons of Science it is a Benefit to the Publick, their Expectations of farther Improvements having then the best Foundation. And All who have the Pleasure of your Acquaintance, and know your Attachment to polite and useful Learning, in which a Knowledge of the Mathematicks may be justly included, will be sensible of my Happiness in being thus permitted to address You.

Believe

5-10-46 ADE

Digitized by Google

Believe me, Sir, whatever may be the Fate of theſe Sheets, I ſhall, at all Times, conſider this Uſe of your Name as a ſingular Honour to,

SIR,

*Your moſt obedient, and moſt*

*Humble Servant,*

T. SIMPSON.

*Woolwich, May* 1, 1752.

Digitized by Google

# THE

# PREFACE.

*THE enſuing Work, or at leaſt the greateſt part of it, was originally compoſed for my own uſe in the* ROYAL ACADEMY: *And it is upon a preſumption that it may alſo be of ſervice to others, eſpecially thoſe employed in a like public way of teaching, that it now appears in the World.*

*The Work itſelf conſiſts of ſix* * *diſtinct parts, or tracts; each of which I ſhall here give ſome account of.*

*The firſt part contains a number of Algebraical Problems, with their Solutions; deſigned as proper exerciſes for young beginners. In the courſe of theſe Problems and Solutions (whereof the greater part will appear to be new) the art of managing equations, and the various methods of ſubſtitution are taught and illuſtrated.*

*The*

---

* The fifth part, relating to the Doctrine of Annuities, is omitted in this Edition, and added to the Author's Treatiſe on that ſubject.

Digitized by Google

*The second part comprehends a variety of Geometrical Problems with their Solutions, both by Algebra and also independent of it, from principles purely Geometrical. In this part the learner will find a large field to exercise his industry in: he will moreover have the opportunity of comparing the two methods of solution together, and from thence observing, that sometimes the one has the advantage, and sometimes the other; that, in some cases they both proceed upon the very same properties, and in others, upon quite different ones: and it may be further remarked from hence (which will be of some use to know) that, when quantities are given in magnitude* only, *the Algebraic method generally claims the preference, in point of ease and expedition, at least; whereas the advantage is almost always on the side of the geometrical effection, when the positions of points and lines, and the quantities of angles are given.*

*There is, however, one particular, or two, in this part, that may be thought to stand in need of some apology.*

*In the first place, the frequent use of symbols, common to the Algebraic notation, may, perhaps, be looked upon as repugnant to the rigour and strictness of Geometry. But it is not the use of symbols (which some, more scrupulous*

*pulous than diſcerning, have condemned) but the ideas annexed to them, that render the conſideration* Geometrical, *or* Ungeometrical. *In pure Geometry regard is always had to the abſolute quantity of ſome one of the three kinds of extenſion, abſtractedly conſidered; and, whatever ſymbols are uſed here, are to be conſidered as expreſſive of the quantities themſelves, and not as any meaſures, or numerical values of them. Thus by* A × B, *taken in a geometrical ſenſe, we have an idea, not of the product of two numbers (as in the algebraic notation), but of a* real, *rectangular, ſpace comprehended under two right-lines, repreſented by* A *and* B, *and two others equal to them. So, likewiſe,* $\frac{B \times C}{A}$ *is not to be underſtood here in the Light of an algebraic fraction, but as a right-line, which is fourth proportional to three other right-lines, repreſented by* A, B *and* C. — *Theſe diſtinctions are abſolutely neceſſary to thoſe who would have an accurate idea of the ſubject.*

*The ſecond particular, above hinted at, relates to the quotations; wherein I have referred to my own Elements of Geometry, and not to thoſe of Euclid, ſo univerſally known and eſtabliſhed. But for this there were two reaſons: Firſt, thoſe perſons, for whoſe inſtruction theſe ſheets are, in a more particular manner,*

Digitized by Google

*manner, deſigned, are taught the firſt principles of Geometry from other Elements than thoſe of Euclid; and, ſecondly, a number of propoſitions are uſed here that are only to be met with in modern authors.*

*In the third part the Theory of Gunnery, or the Motion of Projectiles, is conſidered, excluſive of the conic Sections; and the practical ſolutions of the ſeveral caſes depending on the theory (as well thoſe where the object is elevated or depreſſed as where it is ſituate in the plain of the horizon) are given, at large, by plane Trigonometry.*

*The fourth part exhibits a new, and very comprehenſive method for extracting the roots of algebraical Equations; whereby the number ſought may be determined, to any propoſed degree of exactneſs, without the trouble of repeating the operation, as in the common way, by converging Series's.*

*The fifth part gives ſome account of the Nature of Fluxions, together with the inveſtigation of the fundamental rules; and may be of uſe, not only to beginners, but alſo to ſuch, who, though tolerably well verſed in the practiſe and application of Fluxions, have nevertheleſs but an imperfect idea of the firſt principles of this difficult branch of ſcience.*

May, 1752.

MEMOIRS

Digitized by Google

# MEMOIRS *of the* LIFE *and* WRITINGS *of the* AUTHOR.

MR. THOMAS SIMPSON was born at *Market Bosworth*, in the county of *Leicester*, August the 20th, O. S. 1710. His father was a stuff-weaver in that town; and, though in tolerable circumstances, yet, intending to bring up his son Thomas to his own business, he took so little care of his education, that he was only taught to read English. But nature had furnished him with talents and a genius for very different pursuits, which led him afterwards to the highest rank in the mathematical and philosophical sciences.

Young Simpson very soon gave indications of his turn for study in general, by eagerly reading all books he could meet with, teaching himself to write, and embracing every opportunity he could find of deriving knowledge from other persons. Thomas's father observing him thus to neglect his business, by spending his time in reading what he thought useless books, and following other such like pursuits, used all his endeavours to check such proceedings, and to induce him to follow his profession with steadiness and better effect. But after many struggles for this purpose, the differences thus produced between them at length rose to such a height, that our Author quitted his father's house entirely.

Upon this occasion he repaired to Nuneaton, a town at a small distance from Bosworth, where he went to lodge at the house of a taylor's widow, of the name of Swinfield, who had been left with two children, a daughter and a son, by her husband, of whom the son, who was the younger, being but about two years older than Simpson, had become his intimate friend and companion. And here he continued some

a time

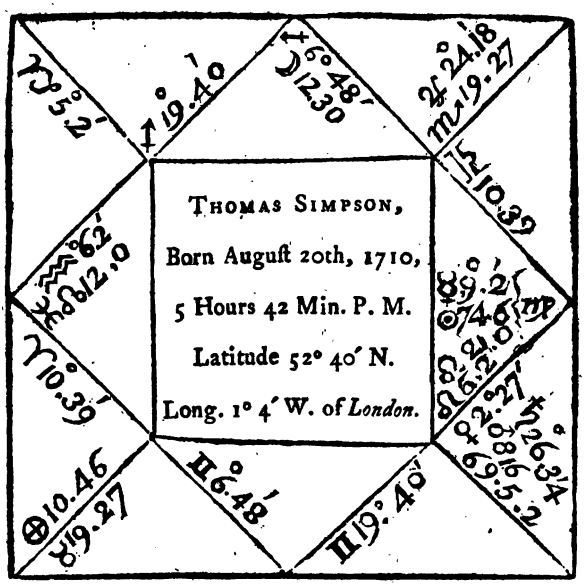
Digitized by Google

time, working at his trade, and improving his knowledge by reading ſuch books as he could procure.

Among ſeveral other circumſtances which, long before this, gave occaſion to ſhew our Author's early thirſt for knowledge, as well as proving a freſh incitement to acquire it, was that of a large ſolar eclipſe, which took place on the 11th of May, 1724. This phenomenon, ſo awful to many who are ignorant of the cauſe of it, ſtruck the mind of young Simpſon with a ſtrong curioſity to diſcover the reaſon of it, and to be able to predict the like ſurpriſing events. It was however five or ſix years before he could obtain his deſire, which at length was gratified by the following accident. After he had been ſome time at Mrs. Swinfield's, at Nuneaton, a travelling pedlar came that way, and took a lodging at the ſame houſe, according to his uſual cuſtom. This man, to his profeſſion of an itinerant merchant, had joined the more profitable one of a fortune-teller, which he performed by means of judicial aſtrology. Every one knows with what regard perſons of ſuch a caſt are treated by the inhabitants of country villages; it cannot be ſurpriſing therefore that an untutored lad of nineteen ſhould look upon this man as a prodigy, and, regarding him in this light, ſhould endeavour to ingratiate himſelf into his favour; in which he ſucceeded ſo well, that the ſage was no leſs taken with the quick natural parts and genius of his new acquaintance. The pedlar, intending a journey to Briſtol fair, left in the hands of young Simpſon an old edition of Cocker's Arithmetic, to which was ſubjoined a ſhort Appendix on Algebra, and a Book upon Genitures, by Partridge the almanac maker. Theſe books he had peruſed to ſo good purpoſe, during the abſence of his friend, as to excite his amazement upon his return; in conſequence of which he ſet himſelf about erecting the following genethliacal type, in order to a preſage of Thomas's future fortune.

THOMAS

Digitized by Google




This poſition of the heavens having been maturely conſidered *ſecundum artem*, the wizard, with great confidence, pronounced, that "within two years time Simpſon would turn out a greater man than himſelf."

In fact, our Author profited ſo well by the encouragement and aſſiſtance of the pedlar, afforded him from time to time when he occaſionally came to Nuneaton, that, by the advice of his friend, he at length made an open profeſſion of caſting nativities himſelf; from which, together with teaching an evening ſchool, he derived a pretty pittance, ſo that he greatly neglected his weaving, to which indeed he had never manifeſted any very great attachment, and ſoon became the oracle of Nuneaton, Boſworth, and the environs. Scarce a courtſhip advanced to a match, or a bargain to a ſale, without previouſly conſulting the infallible Simpſon about the conſequences. But helping folks to ſtolen goods, he always declared above his ſkill; and that as to



Digitized by Google

life and death he had no power: all those called *lawful questions* he readily resolved, provided the persons were certain as to the horary *data* of the horoscope: and, he has often declared, with such success, that if from very cogent reasons he had not been thoroughly convinced of the vain foundation and fallaciousness of his art, he never should have dropt it, as he afterwards found himself in conscience bound to do.

About this time he married the widow Swinfield, in whose house he lodged, though she was then almost old enough to be his grandmother, being upwards of fifty years of age, and having been looked upon as an old maid before her first marriage, and yet her youngest child was two years older than Simpson himself. After this the family lived comfortably enough together for some short time, Simpson occasionally working at his business of a weaver in the day-time, and teaching an evening school or telling fortunes at night; the family being also farther assisted by the labours of young Swinfield, who had been brought up in the profession of his father. But this tranquillity was soon interrupted, and our Author driven at once from his home and the profession of astrology, by the following accident. A young woman in the neighbourhood had long wished to hear or know something of her lover, who had been gone to sea; but Simpson had put her off from time to time, till the girl grew at last so importunate, that he could deny her no longer. He asked her if she would be afraid if he should raise the devil, thinking to deter her; but she declared she feared neither ghost nor devil, so he was obliged to comply. The scene of action pitched upon was a barn, and young Swinfield was to act the devil or ghost, who being concealed under some straw in a corner of the barn, was, at a signal given, to rise slowly out from among the straw, with his face marked so that the girl might not know him. Every thing being in order, the girl came at the time appointed; when Simpson, after cautioning her not to be afraid, began

Digitized by Google

began muttering ſome myſtical words, and chalking round about them, till, on the ſignal given, up riſes the taylor ſlow and ſolemn, to the great terror of the poor girl, who, before ſhe had ſeen half his ſhoulders, fell into violent fits, crying out it was the very image of her lover; and the effect upon her was ſo dreadful, that it was thought either death or madneſs muſt be the conſequence. So that poor Simpſon was obliged immediately to abandon at once both his home and the profeſſion of a conjuror.

Upon this occaſion it would ſeem he fled to Derby, where he remained ſome two or three years, inſtructing pupils in an evening ſchool, and working at his trade by day. Here too he wrote a ſong on occaſion of the parliamentary election at this place, in the year 1733, in favour of the Cavendiſh family; and it is probable he continued here till about the year 1735 or 1736, as I have been informed by the late Mr. Whitehurſt, who then knew our Author, having alſo come to ſettle at Derby about the ſame time with him.

It would ſeem indeed that Simpſon had an early turn for verſifying, both from the circumſtance of the ſong abovementioned, and from his firſt two mathematical queſtions that were publiſhed in the Ladies Diary, which were both in a ſet of verſes, not ill written for the occaſion. Theſe were printed in the Diary for 1736, and therefore muſt at lateſt have been written in the year 1735. Theſe two queſtions, being at that time pretty difficult ones, ſhew the great progreſs he had even then made in the mathematics; and from an expreſſion in the firſt of them, viz. where he mentions his reſidence as being in latitude 52°, it appears he was not then come up to London, though he muſt have done ſo very ſoon after.

Together with his aſtrology he had ſoon furniſhed himſelf with enough of arithmetic, algebra, and geometry to be qualified for looking into the Ladies Diary (of which he had afterwards for ſeveral years the direction), by which he came to underſtand that there

Digitized by Google

there was a ſtill higher branch of the mathematical knowledge than any he had yet been acquainted with; and this was the method of *Fluxions*. But our young analyſt was quite at a loſs to diſcover any Engliſh author who had written on the ſubject, except Mr. Hayes; and his work being a folio, and then pretty ſcarce, exceeded his ability of purchaſing: however, an acquaintance lent him Mr. Stone's *Fluxions*, which is a tranſlation of the *Marquis de l'Hoſpital's Analyſe des Infiniment Petits:* by this one book, and his own penetrating talents, he was, as we ſhall preſently ſee, enabled in a very few years to compoſe a much more accurate treatiſe on this ſubject than any that had before appeared in our language.

After he had quitted aſtrology and its emoluments, he was driven to hardſhips for the ſubſiſtence of his family, while at Derby, notwithſtanding his other induſtrious endeavours in his own trade by day, and teaching pupils at evenings. This determined him to repair to London, which he did in 1735 or 1736, as before ſaid, leaving the family behind, his wife being then pregnant with her firſt child by our Author, which was contrary to the expectation of every one, being then about fifty-five years of age, for which reaſon her neighbours uſed to ſay, "there goes Sarah the wife of Abraham."

On his firſt coming to London, Mr. Simpſon wrought for ſome time at his buſineſs in Spitalfields, and taught mathematics at evenings, or any ſpare hours. His induſtry turned to ſo good account, that he returned down into the country, and brought up his wife and three children, ſhe having produced her firſt child in his abſence. The number of his ſcholars increaſing, and his abilities becoming in ſome meaſure known to the public, he was encouraged to make propoſals for publiſhing by ſubſcription, *A new Treatiſe of Fluxions: wherein the Direct and Inverſe Method are demonſtrated after a new, clear, and conciſe Manner, with their Application to Phyſics and Aſtronomy: alſo the Doctrine of*

Digitized by Google

*of Infinite Series and Reverting Series univerſally, are amply explained, Fluxionary and Exponential Equations ſolved: together with a Variety of new and curious Problems.*

When Mr. Simpſon firſt propoſed his intentions of publiſhing ſuch a work, he did not know of any Engliſh book, founded on the true principles of Fluxions, that contained any thing material, eſpecially the practical part; and though there had been ſome very curious things done by ſeveral learned and ingenious gentlemen, the principles were nevertheleſs left obſcure and defective, and all that had been done by any of them in *infinite ſeries*, very inconſiderable.

The book was publiſhed in 4to, in the year 1737, although the author had been frequently interrupted from furniſhing the preſs ſo faſt as he could have wiſhed, through his unavoidable attention to his pupils for his immediate ſupport. The principles of fluxions treated of in this work, are demonſtrated in a method accurately true and genuine, not eſſentially different from that of their great inventor, being entirely expounded by finite quantities. In the firſt and ſecond parts are given a great many new, and ſome very curious examples in the ſolutions of problems, rendered plain to ordinary capacities.

The ſecond part treats of *Infinite Series*. And here nothing is propoſed without demonſtration, and every thing is illuſtrated by eaſy examples. New rules are alſo laid down for finding the forms of ſeries, without taking in any of the ſuperfluous terms.

The third part contains a familiar method of finding and comparing fluents, illuſtrated with ſome uſeful and eaſy applications.

In the fourth part is ſhewn the uſe of fluxions in ſome of the moſt ſublime branches of *Phyſics* and *Aſtronomy*; where, beſides ſeveral things done in a method quite different from any thing to be met with in other authors, there are ſome very uſeful ſpecu-

lations

Digitized by Google

lations relating to the doctrine of Pendulums and Centripetal Forces.

To this is added a ſupplement; being a collection of miſcellaneous problems, independant of the foregoing four parts; and containing, among other matters, an inveſtigation of the Areas of Spherical Triangles; the Curve of Purſuit; the Paths of Shadows; the Motion of Projectiles in a Medium; and the manner of finding the Attractive Force of bodies of different forms, acting according to a given law.

In 1740, Mr. Simpſon publiſhed a Treatiſe on *The Nature and Laws of Chance*, in 4to. To which is annexed, *Full and clear Inveſtigations of two important Problems added in the 2d Edition of Mr.* De Moivre's *Book on Chances*, as alſo *two New Methods for the Summation of Series.*

Our Author's next publication was a 4to volume of *Eſſays on ſeveral curious and intereſting Subjects in Speculative and Mixed Mathematics*; printed in the ſame year 1740: dedicated to FRANCIS BLAKE, Eſq. ſince Fellow of the Royal Society, and our Author's good friend and patron.

The firſt of theſe Eſſays ſhews the Theory of the apparent Place of the Stars (commonly called their *Aberration)* ariſing from the progreſſive Motion of Light, and of the Earth in its Orbit; with practical Rules for computing the ſame, communicated by Dr. Bevis.

The 2d treats of the Motion of Bodies affected by projectile and centripetal Forces; in which the moſt conſiderable Matters in the firſt Book of Newton's *Principia* are clearly inveſtigated.

The 3d is a Solution of Kepler's Problem, with a conciſe practical Rule.

The 4th treats of the Motion and Paths of Projectiles in reſiſting mediums; determining the moſt important particulars upon this head, in the 2d Book of the *Principia.*

The

Digitized by Google

The 5th conſiders the Reſiſtances, Velocities, and Times of vibration of pendulous Bodies in Mediums.

The 6th contains a new Method of Solution of all Kinds of algebraical equations in numbers, more general than ever before given.

The 7th is concerning the Method of *Increments*, with examples.

The 8th is a conciſe Inveſtigation of a Theorem for finding the Sum of a Series of Quantities, by Means of their Differences.

The 9th is a general Way of inveſtigating the Sum of a recurring Series.

The 10th is a new and general Method for finding the Sum of any Series of Powers, whoſe Roots are in arithmetical Progreſſion; being alſo applicable to Series of other Kinds.

The 11th relates to angular Sections, with ſome remarkable Properties of the circle.

The 12th ſhews an eaſy and expeditious Method of reducing a compound Fraction to ſimple ones.

The 13th, or laſt, which contains a general Quadrature of hyperbolical Curves, is a problem that had exerciſed the ſkill of ſeveral eminent Mathematicians. None of the ſolutions before publiſhed extended farther than to particular caſes, except one, which is in the Philoſophical Tranſactions, without demonſtration, by Mr. Klingenſtierna, profeſſor of mathematics at Upſal. This is inveſtigated by Mr. Simpſon in two different ways; and the general conſtruction is rendered remarkably eaſy, ſimple, and fit for practice. And, on this head it would ſeem that Mr. Klingenſtierna was well pleaſed with what Mr. Simpſon had done; for being afterwards appointed ſecretary to the Royal Academy at Stockholm, as a mark of his eſteem, he procured a diploma, to be tranſmitted to him, by which he was conſtituted a member of that learned body.

Our Author's next work was, *The Doctrine of Annuities and Reverſions, deduced from general and evident*



Digitized by Google

*Principles: with useful Tables, shewing the Values of Single and Joint Lives, &c.* in 8vo, 1742. This was followed, in 1743, by an *Appendix, containing some Remarks on a late Book on the same Subject* (by Mr. Abr. De Moivre, F. R. S.) *with Answers to some personal and malignant Representations in the Preface thereof.* To this answer Mr. De Moivre never thought fit to reply. A new edition of this work has lately been published, augmented with the tract upon the same subject that was printed in our Author's *Select Exercises.*

In 1743 also was published his *Mathematical Dissertations on a Variety of Physical and Analytical Subjects*, in 4to; containing, among other particulars,

A Demonstration of the true Figure which the Earth, or any Planet, must acquire from its rotation about an Axis. A general Investigation of the Attraction at the Surfaces of Bodies nearly spherical. A Determination of the Meridional Parts, and the Lengths of the several Degrees of the Meridian, according to the true Figure of the Earth. An Investigation of the Height of the Tides in the Ocean. A new Theory of Astronomical Refractions, with exact Tables deduced from the same. A new and very exact Method for approximating the Roots of Equations in Numbers; which quintuples the number of Places at each Operation. Several new Methods for the summation of Series. Some new and very useful Improvements in the Inverse Method of Fluxions. The work being dedicated to MARTIN FOLKES, Esq. president of the Royal Society.

His next book was *A Treatise of Algebra, wherein the fundamental Principles are demonstrated, and applied to the solution of a Variety of Problems.* To which he added, *The Construction of a great Number of Geometrical Problems, with the Method of resolving them numerically.*

This work, which was designed for the use of young beginners, was inscribed to WILLIAM JONES, Esq. F. R. S. and printed in 8vo, 1745. And a new edition appeared in 1755, with additions and improvements;

Digitized by Google

ments; among which was a new and general Method of resolving all Biquadratic Equations, that are complete, or having all their terms. This edition was dedicatad to JAMES EARL of MORTON, F. R. S. Mr. JONES being then dead. The work has gone through several other editions since that time: the 6th, or last, was in 1790. *

His next work was, *Elements of Geometry, with their Application to the Mensuration of Superfices and Solids, to the determination of Maxima and Minima, and to the Construction of a great Variety of geometrical Problems:* first published in 1747, in 8vo. And a second edition of the same came out in 1760, with great alterations and additions, being in a manner a new work, designed for young beginners, particularly for the gentlemen educated at the Royal Military Academy at Woolwich, and dedicated to CHARLES FREDERICK, Esq. Surveyor General of the Ordnance. And other editions have appeared since.

Mr. Simpson met with some trouble and vexation in consequence of the first edition of his Geometry. First, from some reflections made upon it, as to the accuracy of certain parts of it, by Dr. Robert Simson, the learned professor of mathematicks in the University of Glasgow, in the notes subjoined to his edition of Euclid's Elements. This brought an answer to those remarks from Mr. Simpson, in the notes added to the 2d edition as above; to some parts of which Dr. Simson again replied in his notes on the next edition of the said Elements of Euclid.

The second was by an illiberal charge of having stolen his Elements from Mr. Muller, the professor of fortification and artillery at the same academy at Woolwich, where our Author was professor of geometry and mathematics. This charge was made at the end of the preface to Mr. Muller's Elements of Mathematics, in two volumes, printed in 1748, in these words:

 "There

* A regular list of the Author's Publications will be given at the end.

Digitized by Google

" There has lately been publiſhed a book under the " title of Elements of Plane Geometry, deſigned for " the Uſe of Schools, which is an incorrect Copy of " the firſt eight Sections of this Work, lent the pre- " tended Author upon a particular Occaſion, and " printed in a ſpurious manner, without my know- " ledge or conſent; an Action too ſcandalous for " any Man of honour to be guilty of. The Editor " imagined, I ſuppoſe, that the changing ſome pro- " poſitions, and mangling the demonſtrations of others, " was a ſufficient Diſguiſe to make it paſs for his own " performance; but how far this will juſtify ſuch a " piece of pyracy, muſt be left to the judgement of " the public."

To this extraordinary charge, Mr. Simpſon, in the preface to the 2d edition, replies, " Were I to attempt " to deſcribe the ideas excited in my mind by the " ſingular modeſty of this important and ſolemn " appeal to the publick, I ſhould be at a loſs for fit " words to expreſs them without tranſgreſſing the " bounds of decency. But I hope that I have not " deſerved ſo ill of the publick, to be thought capable " of acting ſo very humble a part, as that of copying " from *this* author, and of mangling *his* demonſtra- " tions, in order to make them paſs for *my own*.— " That a manuſcript of his (containing between 20 " and 30 of the principal Theorems in Geometry, " extremely ill digeſted) came into my hands, is in- " deed true; but it was not *lent me*, but forced upon " me, by himſelf (the very firſt night after my removal " to Woolwich) in virtue of an *article* in the original " rules and inſtructions for the Academy; whereby it " is ordered, that the ſecond maſter ſhall teach geo- " metry under the direction of the firſt maſter. But " this well intended article, which has been made " ſubſervient to the purpoſes of ignorant tyranny, " and daring calumny, has ſince, in conſequence of " a publick examination, been annulled by an expreſs " order of the Maſter-General of the Ordnance.—I " could mention ſome particulars, ſupported by good " authority,

Digitized by Google

" authority, that occurred in the courſe of that ex-
" amination, which would but ill agree with the im-
" portance he aſſumes in his confident accuſations;
" but I do not think it worth while: This gentleman
" has, himſelf, by his different publications, ſo well
" convinced the world of his abilities, as to render
" any farther comment on that head intirely unne-
" ceſſary and ineffectual." And happy it was for the Academy that the Board of Ordnance relieved the Profeſſor of Mathematicks from any control of that of Fortification, conſidering what ſort of profeſſors have ſucceeded Mr. Muller, and probably may again; for whatever he might be in other reſpects, he was not altogether unacquainted with the mathematical ſciences.

In 1748 came out Mr. Simpſon's *Trigonometry, Plane and Spherical, with the Conſtruction and Application of Logarithms*, 8vo. This little book contains ſeveral things new and uſeful.

In 1750 came out, in two volumes, 8vo, *The Doctrine and Application of Fluxions, containing, beſides what is common on the Subject, a Number of new Improvements in the Theory, and the Solution of a Variety of new and very intereſting Problems in different Branches of the Mathematics.*—In the preface, the author offers this to the world as a new book, rather than a ſecond edition of that which was publiſhed in 1737, in which he acknowledges, that, beſides errors of the preſs, there are ſeveral obſcurities and defects, for want of experience, and the many diſadvantages he then laboured under, in his firſt ſally.

The idea and explanation here given of the firſt principles of Fluxions, are not eſſentially different from what they are in his former treatiſe, though expreſſed in other terms. The conſideration of *time* introduced into the general definition, will, he ſays, perhaps be diſliked by thoſe who would have fluxions to be *mere velocities*: but the advantage of conſidering them otherwiſe, viz. not as the velocities themſelves, but as magnitudes they would uniformly generate in a given

Digitized by Google

a given time, appears to obviate any objection on that head. By taking fluxions as mere velocities, the imagination is confined as it were to a point, and without proper care insensibly involved in metyphysical difficulties. But according to this other mode of explaining the matter, less caution in the learner is necessary, and the higher orders of fluxions are rendered much more easy and intelligible. Besides, though Sir Isaac Newton defines fluxions to be the velocities of motions, yet he has recourse to the increments or moments generated in equal particles of time, in order to determine those velocities; which he afterwards teaches to expound by finite magnitudes of other kinds. This work was dedicated to GEORGE EARL of MACCLESFIELD.

In 1752 appeared, in 8vo, the *Select Exercises for young Proficients in the Mathematics*. This neat volume contains, A great Variety of algebraical Problems, with their Solutions. A select Number of Geometrical Problems, with their Solutions, both algebraical and geometrical. The Theory of Gunnery, independant of the Conic Sections. A new and very comprehensive Method for finding the Roots of Equations in Numbers. A short Account of the first Principles of Fluxions. Also the Valuation of Annuities for single and joint Lives, with a Set of new Tables, far more extensive than any extant. This last part was designed as a supplement to his Doctrine of Annuities and Reversions; but being thought too small to be published alone, it was inserted here at the end of the Select Exercises; from whence however it has been removed in the last editions, and referred to its proper place, the end of the Annuities, as before mentioned. The examples that are given to each problem in this last piece, are according to the London bills of mortality; but the solutions are general, and may be applied with equal facility and advantage to any other table of observations. The volume is dedicated to JOHN BACON, Esq. F. R. S.

Mr.

Digitized by Google

Mr. Simpſon's *Miſcellaneous Tracts*, printed in 4to, 1757, was his laſt legacy to the public: a moſt valuable bequeſt, whether we conſider the dignity and importance of the ſubjects, or his ſublime and accurate manner of treating them.

The firſt of theſe papers is concerned in determining the *Preceſſion of the Equinox, and the different Motions of the Earth's Axis, ariſing from the Attraction of the Sun and Moon.* It was drawn up about the year 1752, in conſequence of another on the ſame ſubject, by M. de Sylvabelle, a French gentleman. Though this gentleman had gone through one part of the ſubject with ſucceſs and perſpicuity, and his concluſions were perfectly conformable to Dr. Bradley's obſervations; it nevertheleſs appeared to Mr. Simpſon that he had greatly failed in a very material part, and that indeed the only very difficult one; that is, in the determination of the momentary alteration of the poſition of the earth's axis, cauſed by the forces of the ſun and moon; of which forces, the quantities, but not the effects, are truly inveſtigated. The ſecond paper contains the Inveſtigation of a very exact Method or Rule for finding the Place of a Planet in its Orbit, from a correction of Biſhop Ward's circular Hypotheſis, by Means of certain Equations applied to the Motion about the upper Focus of the Ellipſe. By this Method the Reſult, even in the Orbit of Mercury, may be found within a Second of the Truth, and that without repeating the Operation. The third ſhews the Manner of transferring the Motion of a Comet from a parabolic Orbit, to an elliptic one; being of great Uſe, when the obſerved Places of a (new) Comet are found to differ ſenſibly from thoſe computed on the Hypotheſis of a parabolic Orbit. The fourth is an Attempt to ſhew, from mathematical Principles, the Advantage ariſing from taking the Mean of a Number of Obſervations, in practical Aſtronomy; wherein the Odds that the Reſult in this Way, is more exact than from one ſingle Obſervation, is evinced

Digitized by Google

evinced, and the Utility of the Method in Practice, clearly made appear. The fifth contains the Determination of certain Fluents, and the Reſolution of ſome very uſeful Equations, in the higher Orders of Fluxions, by Means of the Meaſures of Angles and Ratios, and the right and verſed Sines of circular Arcs. The 6th treats of the Reſolution of algebraical Equations, by the Method of Surd-diviſors; in which the Grounds of that Method, as laid down by Sir Iſaac Newton, are inveſtigated and explained. The 7th exhibits the Inveſtigation of a general Rule for the Reſolution of Iſoperimetrical Problems of all Orders, with ſome Examples of the Uſe and Application of the ſaid Rule. The 8th, or laſt part, comprehends the Reſolution of ſome general and very important Problems in Mechanics and Phyſical Aſtronomy; in which, among other Things, the principal Parts of the 3d and 9th Sections of the firſt Book of Sir Iſaac Newton's Principia, are demonſtrated in a new and conciſe Manner. But what may perhaps beſt recommend this excellent tract, is the application of the general equations, thus derived, to the determination of the Lunar Orbit.

According to what Mr. Simpſon had intimated at the concluſion of his Doctrine of Fluxions, the greateſt part of this arduous undertaking was drawn up in the year 1750. About that time M. Clairaut, a very eminent mathematician of the French Academy, had ſtarted an objection againſt Sir Iſaac Newton's general law of gravitation. This was a motive to induce Mr. Simpſon (among ſome others) to endeavour to diſcover whether the motion of the moon's apogee, on which that objection had its whole weight and foundation, could not be truly accounted for, without ſuppoſing a change in the received law of gravitation, from the inverſe ratio of the ſquares of the diſtances. The ſucceſs anſwered his hopes, and induced him to look farther into other parts of the theory of the moon's motion, than he had at firſt intended:

Digitized by Google

intended: but before he had completed his design, M. Clairaut arrived in England, and made Mr. Simpson a visit; from whom he learnt, that he had a little before printed a piece on that subject, a copy of which Mr. Simpson afterwards received as a present, and found in it the same things demonstrated, to which he himself had directed his enquiry, besides several others.

The facility of the method Mr. Simpson fell upon, and the extensiveness of it, will in some measure appear from this, that it not only determines the motion of the apogee, in the same manner, and with the same ease, as the other equations, but utterly excludes all that dangerous kind of terms that had embarrassed the greatest mathematicians, and would, after a great number of revolutions, entirely change the figure of the moon's orbit. From whence this important consequence is derived, that the moon's mean motion, and the greatest quantities of the several equations, will remain unchanged, unless disturbed by the intervention of some foreign or accidental cause. These tracts are inscribed to the EARL of MACCLESFIELD, President of the Royal Society.

Besides the foregoing, which are the whole of the regular books or treatises that were published by Mr. Simpson, he wrote and composed several other papers and fugitive pieces, as follow.

Several papers of his were read at the meetings of the Royal Society, and printed in their Transactions: but as most, if not all of them, were afterwards inserted, with alterations or additions, in his printed volumes, it is needless to take any farther notice of them here.

He proposed, and resolved many questions in the Ladies Diaries, &c. sometimes under his own name, as in the years 1735 and 1736; and sometimes under feigned or fictitious names; such as, it is thought, Hurlothrumbo, Kubernetes, Patrick O'Cavenah, Marmaduke Hodgson, Anthony Shallow, Esq. and probably several others; see the Diaries for the years 1735,



Digitized by Google

1736, 42, 43, 53, 54, 55, 56, 57, 58, 59, and 60. Mr. Simpſon was alſo the editor or compiler of the Diaries from the year 1754 till the year 1760, both incluſive, during which time he raiſed that work to the greateſt degree of reſpect. He was ſucceeded in the editorſhip by Mr. Edw. Rollinſon. See my Diarian Miſcellany, vol. 3.

It has alſo been commonly ſuppoſed that he was the real editor of, or had a principal ſhare in, two other periodical works of a miſcellaneous mathematical nature; viz. the Mathematician, and Turner's Mathematical Exerciſes, two volumes, in 8vo, which came out in periodical numbers, in the years 1750 and 1751, &c. The latter of theſe ſeems eſpecially to have been ſet on foot to afford a proper place for expoſing the errors and abſurdities of Mr. Robert Heath, the then conductor of the Ladies Diary and the Palladium; and which controverſy between them ended in the diſgrace of Mr. Heath, and expulſion from his office of editor to the Ladies Diary, and the ſubſtitution of Mr. Simpſon in his ſtead, in the year 1753.

In the year 1760, when the plans propoſed for erecting a new bridge at Blackfriars were in agitation, Mr. Simpſon, among other gentlemen, was conſulted upon the beſt form for the arches, by the New-bridge Committee. Upon this occaſion he gave a preference to the ſemicircular form; and, beſides his report to the Committee, ſome letters alſo appeared, by himſelf and others, on the ſame ſubject, in the public newspapers, particularly in the Daily Advertiſer, and in Lloyd's Evening Poſt. And the ſame were alſo collected in the Gentleman's Magazine for that year, page 143 and 144.

It is probable that this reference to him, gave occaſion to the turning his thoughts more ſeriouſly to this ſubject, ſo as to form the deſign of compoſing a regular treatiſe upon it: for his family have often informed me that he laboured hard upon this work for ſome time before his death, and was very anxious to have completed it, frequently remarking to them, that this

Digitized by Google

this work, when publiſhed, would procure him more credit than any of his former publications. But he lived not to put the finiſhing hand to it. Whatever he wrote upon this ſubject, probably fell, together with all his other remaining papers, into the hands of Major Henry Watſon, of the Engineers, in the ſervice of the India Company, being in all a large cheſt full of papers. This gentleman had been a pupil of Mr. Simpſon's, and had lodged in his houſe. After Mr. Simpſon's death, Mr. Watſon prevailed upon the widow to let him have the papers, promiſing either to give her a ſum of money for them, or elſe to print and publiſh them for her benefit. But nothing of the kind was ever done; this gentleman always declaring, when urged on this point by myſelf and others, that no uſe could be made of any of the papers, owing to the very imperfect ſtate in which he ſaid they were left. And yet he perſiſted in his refuſal to give them up again.

From Mr. Simpſon's writings, I now return to himſelf. Through the intereſt and ſolicitations of the beforementioned William Jones, Eſq. he was, in 1743, appointed profeſſor of mathematics, then vacant by the death of Mr. Derham, in the Royal Academy at Woolwich; his warrant bearing date Auguſt 25th. And in 1745 he was admitted a fellow of the Royal Society, having been propoſed as a candidate by Martin Folkes, Eſq. Preſident, William Jones, Eſq. Mr. George Graham, and Mr. John Machin, Secretary; all very eminent mathematicians. The preſident and council, in conſideration of his very moderate circumſtances, were pleaſed to excuſe his admiſſion fees, and likewiſe his giving bond for the ſettled future payments.

At the academy he exerted his faculties to the utmoſt, in inſtructing the pupils who were the immediate objects of his duty, as well as others, whom the ſuperior officers of the ordnance permitted to be boarded and lodged in his houſe. In his manner of teaching he had a peculiar and happy addreſs; a certain dignity and perſpicuity, tempered with ſuch a



Digitized by Google

degree of mildneſs, as engaged both the attention, eſteem and friendſhip of his ſcholars; of which the good of the ſervice, as well as of the community, was a neceſſary conſequence.

It muſt be acknowledged however, that his mildneſs and eaſineſs of temper, united with a more inactive ſtate of mind, in the latter years of his life, rendered his ſervices leſs uſeful; and the ſame very eaſy diſpoſition, with an innocent, unſuſpecting ſimplicity, and playfulneſs of mind, rendered him often the dupe of the little tricks of his pupils. Having diſcovered that he was fond of liſtening to little amuſing ſtories, they took care to furniſh themſelves with a ſtock; ſo that, having neglected to learn their leſſons perfect, they would get round him in a croud, and, inſtead of demonſtrating a propoſition, would amuſe him with ſome comical ſtory, at which he would laugh and ſhake very heartily, eſpecially if it were tinctured with ſomewhat of the ludicrous or ſmutty; by which device they would contrive imperceptibly to wear out the hours allotted for inſtruction, and ſo avoid the trouble of learning and repeating their leſſon. They tell alſo of various tricks that were practiſed upon him in conſequence of the loſs of his memory in a great degree, in the latter ſtage of his life.

Indeed it may ſeem ſtrange that a tall, large, and ſeemingly ſtrong-bodied man (for ſuch was the appearance and figure of Mr. Simpſon), ſhould at fifty years old, without ſickneſs, have all his faculties quite worn out, and die of extreme old age. But, notwithſtanding his bulk and figure, he was really of no very ſtrong conſtitution of body, having originally been framed with weak nerves: beſides which, there were two other principal cauſes for his premature decay: the firſt of theſe was his very cloſe and uninterrupted application to ſtudy, almoſt continually by night and by day, which produced an extraordinary exhauſting of his ſpirits, without allowing him time to relieve and recruit them by proper air and exerciſe, and to enjoy the converſation of his friends: the other was the want of proper nouriſhment, and the cruelty

Digitized by Google

cruelty of a waſteful wife and family, whoſe extravagance kept him always poor and drudging to ſupply their voracious demands, and withheld from him the nouriſhment proper for his comfortable exiſtence. Indeed the many ſtories I have been told of her ill temper, and barbarous treatment of a man ſo gentle, mild and placid, as Mr. Simpſon was, are quite ſhocking, by which his very uſeful and valuable life was doubtleſs ſhortened by many years: while ſhe was revelling with her family and goſſips in the parlour, he was baniſhed to the garret, to toil at his book, with no other ſupper than the thin milk from which ſhe had ſkimmed the cream for her tea, and accompanied with an order that he ſhould not be ſeen below for the night. She would break open his cupboard to rob him of his cordial, and pick his pockets of the trifle he had ſaved to ſpend with his friends; and numberleſs other unnatural actions.

It has been ſaid that Mr. Simpſon frequented low company, with whom he uſed to guzzle porter and gin: but it muſt be obſerved that the miſconduct of his family put it out of his power to keep the company of gentlemen, as well as to procure better liquor.

In the latter ſtage of his exiſtence, when his life was in danger, exerciſe and a proper regimen were preſcribed him, but to little purpoſe; for he ſunk gradually into ſuch a lowneſs of ſpirits, as often in a manner deprived him of his mental faculties, and at laſt rendered him incapable of performing his duty, or even of reading the letters of his friends; and ſo trifling an accident as the dropping of a tea-cup would flurry him as much as if a houſe had tumbled down.

The phyſicians adviſed his native air for his recovery; and in February, 1761, he ſet out, with much reluctance (believing he ſhould never return) for Boſworth, along with ſome relations. The journey fatigued him to ſuch a degree, that upon his arrival he betook himſelf to his chamber, where he grew continually worſe and worſe, to the day of his death, which happened the 14th of May, in the fifty-firſt year of his age.

Upon

Digitized by Google

Upon this occaſion a reflection forces itſelf upon me, concerning the inſalubrity of the ſituation and reſidence of the profeſſor of mathematics at Woolwich, and the ſimilarity of its effects upon the whole ſucceſſion of profeſſors, Mr. Simpſon, Mr. John Lodge Cowley, and myſelf, who every one of us, each after the other, entered upon our appointments at an early age, and with the beſt proſpects of health and long life, that a vigorous conſtitution, founded by a virtuous education at a diſtance from the capital, could promiſe. Yet, ſuch is the cloſeneſs of the profeſſor's apartments, and the almoſt impoſſibility of his taking proper air and exerciſe, joined to the natural ſedentary habits of a ſtudious profeſſion, that it has uniformly reduced us all in a ſimilar manner to the brink of the grave, and nearly in the ſame ſpace of time. Mr. Simpſon was gradually reduced by it to a condition too low before he changed his ſituation, and ended his valuable labours after ſeventeen years reſidence. Mr. Cowley, after twelve years, was ſo reduced, as not to be able to reſide and perform his duty any longer; when, being ſuperannuated on a penſion, he retired to a ſituation of better air, and where he could eaſily have proper exerciſe; which preſently had the beſt effect in reſtoring his health, and he is now, after almoſt twenty years, in as healthy and robuſt a condition as any man whatever. In like manner, ſucceeding Mr. Cowley, I began my career with a cloſe application to ſtudy, to which I was induced both by inclination, and the want of company and exerciſe: but ſoon found an alteration in my health, which gradually declined, till, at the end of thirteen years, I was unable to continue my reſidence any longer, and obliged to retire to a more elevated ſituation, at a ſmall diſtance, from whence, by walking always to the academy and back again, my health and ſtrength gradually improved; and as this experiment evinced the benefit of the change I had been compelled to make, his Grace the Maſter General of the ordnance, and the principal officers of that board, have been pleaſed to permit my reſidence in the like ſituation ever

Digitized by Google

ever ſince, which is now more than ſix years, and with the ſame manifeſt benefit to my health and activity.

But to return to our Author. At his death, Mr. Simpſon left in being, beſides his widow, her daughter and ſon by her firſt huſband, and his own daughter and ſon, the latter of whom was born after his removal to London. The two daughters had married officers of artillery, and young Simpſon had been preſented with a commiſſion in the ſame corps; but all or moſt of them ſpending and ending their lives in a manner that might be expected from an education under ſuch a mother: young Simpſon lived to have a company in the Royal Artillery; but on account of a beſotted habit of drunkenneſs, and other miſconduct, he was broke; and ſeveral years after my ſettlement at Woolwich, he was wandering about as a vagabond, and a ſcandal to all that knew him: having not heard of him for ſeveral years, I ſuppoſe he has died a miſerable death. The daughters likewiſe are both dead. As to Mr. Simpſon's widow, the King, at the inſtances of Lord Viſcount Ligonier, the then Maſter General, in conſideration of Mr. Simpſon's great merits, was graciouſly pleaſed to grant her a handſome penſion for life, together with apartments adjoining to the academy; a favour that had never been conferred on any before. Here ſhe lived, and her ſon Swinfield, the taylor, and his wife, along with her, in the ſame apartments, for many years after my ſettling at Woolwich; and from one or other of whom I have at different times been informed of the particulars above related, with many others. She died only December the 14th, 1782, at the great age of one hundred and two. Her ſon Swinfield is ſtill living, and reſides in Woolwich, ſupported by a weekly allowance from ſome relations, who are in very genteel circumſtances; he is now about eighty-four years of age, and looks ſo well, that he may probably ſtill exiſt ſeveral years longer.

*Royal Academy, Woolwich,* C. H.
*July* 28, 1792.

Digitized by Google

The following BOOKS are all written by Mr. THOMAS SIMPSON, F. R. S. and printed for F. WINGRAVE, Succeſſor to Mr. NOURSE, in the STRAND.

1. The Elements of Geometry; with their Application to the Menſuration of Superfices and Solids; to the Determination of the Maxima and Minima of Geometrical Quantities, and to the Conſtruction of a great Variety of Geometrical Problems, 8vo, fourth edit. 5s.

2. Trigonometry, Plane and Spherical, with the Conſtruction and Application to Logarithms, third edit. 8vo, 1s 6d.

3. A Treatiſe of Algebra: wherein the fundamental Principles are fully and clearly demonſtrated, and applied to the Solution of a great Variety of Problems, and to a Number of other uſeful Enquiries, fixth edit. 8vo, 6s.

4. The Doctrine and Application of Fluxions; containing (beſides what is common on the Subject) a Number of new Improvements in the Theory, and the Solution of a Variety of new and very intereſting Problems in different Branches of the Mathematics, 2 vols, 8vo, ſecond edit. 12s.

5. Select Exerciſes for young Proficients in the Mathematics, 8vo, new edit. 6s.

6. Eſſays on ſeveral curious and uſeful Subjects in ſpeculative and mixed Mathematics; in which the moſt difficult Problems of the Firſt and Second Books of Newton's Principia are explained, 4to, 6s.

7. Mathematical Diſſertations on a Variety of phyſical and analytical Subjects, 4to, 7s.

8. Miſcellaneous Tracts on ſome curious and very intereſting Subjects in Mechanics, Phyſical Aſtronomy, and ſpeculative Mathematics, 4to, 7s.

9. The Doctrine of Annuities and Reverſions, deduced from general and evident Principles; with uſeful Tables, ſhewing the Values of ſingle and joint Lives, &c. ſecond edit 8vo. 3s.

10. A Supplement to the above, which contains all the learned Author has written on this Subject, 8vo, new edit. 2s 6d.

11. The Nature and Laws of Chance, containing the Solutions of a great Number of abſtruſe and important Problems, after a new, general, and conſpicuous Manner; and illuſtrated with a great Variety of Examples, a new edit. 8vo. 3s.

PART

Digitized by Google

# PART I.

## CONTAINING

*A ſelect Number of* ALGEBRAICAL PROBLEMS, *with their* SOLUTIONS.

## DESIGNED

*As proper Exerciſes for young Beginners.*

---

### QUESTION I.

*WHAT Number is that, which being doubled and* 16 *added to the Product, the Sum ſhall be* 188?

Let $x$ repreſent the required number;
then $2x$ will denote the double thereof;
and ſo $2x+16=188$, *by the queſtion.*
Therefore $2x=188-16=172$, *by tranſpoſition.*
And $x=\frac{172}{2}=86$, *by diviſion.*

### QUESTION II.

*To find that Number, which being added to* 56, *the Treble of the required Number ſhall be produced.*

If $x$ be put for the number ſought,
then $3x$ will be the treble thereof:
and therefore $3x=x+56$, *by the queſtion.*
Hence $2x=56$, *by tranſpoſition.*
And $x=\frac{56}{2}=28$, *by diviſion.*



Digitized by Google

## QUESTION III.

*The Sum of* 155*l. was raiſed (for a certain Purpoſe) by three different Perſons,* A, B, *and* C; *whereof* B *advanced* 15*l. more than* A; *and* C, 20*l. more than* B: *How much did each contribute?*

Let $x$ be the number of pounds advanced by A.
Then $x+15$ is the number of pounds advanced by B,
and $x+35$ the number of pounds advanced by C.
Therefore $3x+50=155$, *by the queſtion.*
Whence $3x=105$,
and $x=35$.
From which it alſo appears that B contributed 50*l.* and C 70*l.*

## QUESTION IV.

*A Gentleman (by Will) left* 550*l. to be divided among four Servants,* A, B, C, *and* D; *whereof* B *was to have twice as much as* A; C *as much as* A *and* B; *and* D *as much as* C *and* B. *How much had each?*

Let $x$ be the number of A's pounds.
Then $2x$ is the number of B's pounds,
alſo $3x$ is the number of C's pounds,
and $5x$ the number of D's pounds:
therefore $11x=550$, *by queſtion.*
And, conſequently, $x=\frac{550}{11}=50$.
From which the reſt of the ſhares are eaſily determined.

## QUESTION V.

*It is required to divide the Number* 92 *into four ſuch Parts, that the firſt may exceed the ſecond by* 10, *the third by* 18, *and the fourth by* 24.

Let $x$ be the firſt part.
Then $\left\{\begin{matrix} x-10 \\ x-18 \\ x-24 \end{matrix}\right\}$ will be the other parts.

And

And $4x-52=92$, *by the queſtion.*
Hence $4x=144$;
and $x=\frac{144}{4}=36$.

## QUESTION VI.

*A certain Sum of Money was ſhared among five Perſons,* A, B, C, D, *and* E; *whereof* B *received* 10*l. leſs than* A; C 16*l. more than* B; D 5*l. leſs than* C; *and* E 15*l. more than* D. *Moreover it appeared that* E *received as much as both* A *and* B. *What was the whole Sum ſhared, and how much did each receive?*

Let $x$ be the ſhare of A.

Then $\left\{\begin{matrix} x-10 \\ x+6 \\ x+1 \\ x+16 \end{matrix}\right\}$ will be that of $\left\{\begin{matrix} \text{B,} \\ \text{C,} \\ \text{D,} \\ \text{E:} \end{matrix}\right.$

therefore $x+16=2x-10$, *by the queſtion.*
Whence $26=x$.
From which it appears that 26, 16, 32, 27, and 42 (pounds) were the reſpective ſhares; and that the whole ſum was 143*l.*

## QUESTION VII.

*To find that Number, whereof the Double increaſed by* 24, *ſhall as much exceed* 80, *as the Number itſelf is below* 100.

Let $x$ be the required number.
Then $2x+24-80=100-x$, *by the queſtion.*
Whence $2x+x=100-24+80$,
that is $3x=156$;
and therefore $x=\frac{156}{3}=52$.



Digitized by Google

## QUESTION VIII.

*What two Numbers are those, whereof the Difference is* 7 *and the Sum* 33.

Let $x$ be the lesser number,
then $x+7$ will be the greater,
and $2x+7=33$.
Therefore $2x=33-7=26$,
and $x=\frac{26}{2}=13=$ the lesser.
Whence $x+7=20=$ the greater.

## QUESTION IX.

*To divide the Number* 75 *into two such Parts, that* 3 *times the greater may exceed* 7 *times the lesser by* 15.

If $x$ be the greater part,

then $75-x$ will be the lesser } *by the question.*
and $3x=\overline{75-x}\times 7+15$
that is, $3x=525-7x+15$;
therefore $10x=540$, and $x=54$.
From whence the lesser part $(75-x)$ is found $=21$.

## QUESTION X.

A, *after winning* 10 *Guineas of* B, *had as much Money as* B *and* 6 *Guineas more*; *and betwixt them both they had forty Guineas. What Money had each at first?*

Let $x$ be the guineas that A began with;
then $40-x$, are the guineas that B began with:
therefore, after play,
A had, $x+10$ guineas;
and B, $30-x$ guineas.
Whence $x+10=30-x+6$ *(by the question.)*
Therefore $2x=26$.
And $x=13$.

QUESTION

Digitized by Google

## QUESTION XI.

*The Sum of* 500*l. was divided among four Perſons, ſo that the firſt and ſecond, between them, had* 280*l; the firſt and third* 260*l; and the firſt ond fourth* 220*l. How many Pounds had Each?*

If $x$ be the number of pounds the firſt had,

then the $\left\{\begin{matrix}2\text{d}\\3\text{d}\\4\text{th}\end{matrix}\right\}$ had $\left\{\begin{matrix}280-x.\\260-x.\\220-x.\end{matrix}\right.$

The ſum of all which,
being $760-2x$, is $=500$, *by the queſtion.*
Whence $x=\frac{760-500}{2}=130$.
Therefore the four ſhares were 130, 150, 130, and 90 pounds, reſpectively.

## QUESTION XII.

*'Tis propoſed to divide* 60 *into two ſuch parts, that the Difference between the Greater and* 64, *may be equal to twice the Difference between the Leſſer and* 38.

If $x$ be the greater part;
then $60-x$ will be the leſſer:
alſo $64-x$ will be the firſt mentioned difference,
and $38-60+x$, or $x-22$, will be the ſecond.
Therefore $64-x=2\times x-22$, *by the queſtion.*
that is, $64-x=2x-44$.
Whence $108=3x$, and $36=x$.

## QUESTION XIII.

*After* 34 *Gallons had been drawn out of one of two, equal, Caſks, and* 80 *Gallons out of the other, there remained juſt twice as much Liquor in the one as in the other. What did each Caſk contain when full?*

Let $x$ be the number of gallons ſought;
then $x-34$ will be what remained in the firſt caſk,

and

Digitized by Google

and $x-80$, what remained in the second,
Hence $x-34=2\times\overline{x-80}$, *by the question.*
Or, $x-34=2x-160$.
Therefore $126=x$.

## QUESTION XIV.

*A Son asking his Father how old he was, received the following Answer. Your Age four Years ago, says the Father, was only $\frac{1}{4}$ of mine, at that Time; but now your Age is just $\frac{1}{3}$ of mine. What was the Age of each?*

Let $x$ be the age of the son,
then $3x$ will be that of the father:
also $x-4$ will be the age of the son four years before the time in question; and $3x-4$ will be the corresponding age of the father: which, *by the question*, is equal to 4 times $x-4$. Hence we have this equation $4x-16=3x-4$.
Therefore $x=12$, and $3x=36$; which are the two ages required.

## QUESTION XV.

*What Number is that, whose $\frac{1}{3}$ exceeds its $\frac{1}{4}$ Part by 16?*

Let $x$ be the required number; then, its $\frac{1}{3}$ part being $\frac{x}{3}$, and its $\frac{1}{4}$ part $\frac{x}{4}$,
we have $\frac{x}{3}-\frac{x}{4}=16$, *by the question.*
Hence $4x-3x=192$, *by reduction.*
that is, $x=192$.

QUESTION

Digitized by Google

## QUESTION XVI.

*In a Mixture of Wine and Cyder, one half of the whole +25 Gallons was Wine; and $\frac{1}{3}$ Part −5 Gallons Cyder. How many Gallons were there of each?*

If $x$ be put for the number of gallons in the whole mixture, the gallons of wine will be expreſſed by $\frac{x}{2}+25$, and thoſe of cyder by $\frac{x}{3}-5$: which together being equal to the *whole*, we

therefore have $\frac{x}{2}+25+\frac{x}{3}-5=x$;

or, $20=x-\frac{x}{2}-\frac{x}{3}$:

Hence $120=6x-3x-2x$;
or $120=x$. From which it appears that the mixture conſiſted of 85 gallons of wine, and 35 of cyder.

## QUESTION XVII.

*In a Lottery conſiſting of 100000 Tickets, half the Number of Prizes added to $\frac{1}{3}$ of the Number of Blanks, was 35000. How many Prizes were there in the Lottery?*

If $x$ be the number of prizes;
then $100000-x$, will be the number of blanks.
And ſo, $\frac{x}{2}+\frac{100000-x}{3}=35000$.
Hence $3x+200000-2x=210000$:
and $x=10000=$ the number ſought.

QUESTION

## QUESTION XVIII.

*To the Compoſition of a certain Quantity of Gunpowder, $\frac{1}{2}$ of the whole Weight +6lb. of Saltpetre was neceſſary; the Sulphur uſed was $\frac{1}{3}$ of the whole −5lb. and the Charcoal $\frac{1}{4}$ of the whole −3lb. How many Pounds of each of the three Ingredients were there taken?*

Let $x$ be the number of pounds in the whole: then

there were $\begin{cases} \frac{x}{2}+6, \text{ pounds of ſaltpetre.} \\ \frac{x}{3}-5, \text{ pounds of ſulphur.} \\ \frac{x}{4}-3, \text{ pounds of charcoal.} \end{cases}$

And therefore $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}-2=x$, *by the queſtion.*

Whence $12x+8x+6x-48=24x$:

And conſequently $x=\frac{48}{2}=24$.

Therefore there were taken 18lb. of ſaltpetre, 3lb. of ſulphur, and 3lb. of charcoal.

## QUESTION XIX.

*A General, after having loſt a Battle, found that he had only $\frac{1}{2}$ of his Army +3600 left, fit for Action; $\frac{1}{8}$ of his Men +600 being wounded, and the Reſt, which were $\frac{1}{5}$ of the whole Army, either ſlain, taken Priſoners, or miſſing. What was the whole Number of the Army?*

The number ſought being denoted by $x$, the number of men left unhurt will be $\frac{x}{2}+3600$.

And the number of the wounded $\frac{x}{8}+600$.

To which adding, $\frac{x}{5}$, the number of the ſlain and priſoners, we have the number of the whole army;

or

Digitized by Google

or $\frac{x}{2}+3600+\frac{x}{8}+600+\frac{x}{5}=x$.

Hence $4200\ (=x-\frac{x}{2}-\frac{x}{8}-\frac{x}{5})=\frac{3x}{8}-\frac{x}{5}$.

Therefore $168000\ (=15x-8x)=7x$:

and $24000=x$.

## QUESTION XX.

*A Prize of* 2000*l. was divided between two Perſons,* A *and* B; *whoſe Shares therein were in proportion as* 7 *to* 9. *What was the Share of Each?*

Let $x$ denote the ſhare of A;

then $2000-x$, will be that of B.

But $x : 2000-x :: 7 : 9$, *by the queſtion.*

From whence (as the rectangle of the two extremes, of any four proportional numbers, is equal to the rectangle of the two means) we get this equation,

$9\times x=2000-x\times 7$;

that is, $9x=14000-7x$.

Hence $16x=14000$,

and $x=875$.

Therefore the ſhare of A was 875*l.* and that of B, 1125*l.*

## QUESTION XXI.

*To divide* 44 *into two ſuch Parts, that the Greater increaſed by* 5, *may be to the Leſſer increaſed by* 7, *as* 4 *is to* 3.

If $x$ be the greater part,

$44-x$, will be the leſſer,

and $x+5 : 51-x :: 4 : 3$, *by the queſtion.*

Therefore (by multiplying extremes and means) we have

$3x+15=204-4x$:

Whence $x=\frac{204-15}{7}=\frac{189}{7}=27$.



Digitized by Google

## QUESTION XXII.

*To find two Numbers in the proportion of* 1 *to* 2; *ſo that,* 12 *being added to Each, the Sums ſhall be in proportion as* 5 *is to* 7.

Let $x$ be the leſſer number;
then $2x$, will be the greater.
Hence $x+12 : 2x+12 :: 5 : 7$, *by the queſtion.*
Therefore $2x+12\times 5=x+12\times 7$;
that is $10x+60=7x+84$.
From which $3x=24$; and $x=8$.
So that the two numbers are 8 and 16.

## QUESTION XXIII.

A, *at Play, firſt won* 5 *Guineas of* B, *and had then as much Money as* B; *but* B, *upon winning back his own Money and* 5 *Guineas more, had* 5 *times as much Money as* A. *What Money had each at firſt?*

If $x$ be the number of A's guineas, at firſt; then it is plain, from the queſtion, that $x+10$ will be the number of B's guineas.

Whence $x+10+5\ (=x-5\times 5)=5x-25$,
that is, $40=4x$.
Therefore $x=10=$ the number of A's guineas;
and $x+10=20=$ the number of B's guineas.

## QUESTION XXIV.

*A Grocer, with* 56*lb. of fine Tea, at* 20 *Shillings a Pound, would mix a coarſer Sort, of* 14 *Shillings, ſo as to afford the whole, together, at* 18 *Shillings per Pound. What Quantity of the latter Sort muſt he take?*

Let $x$ be the number of pounds required;
then 1120 is the value of the fineſt ſort,
and $14x$ that of the coarſeſt.

Moreover,

Digitized by Google

Moreover, the number of pounds of both ſorts together being $56+x$, it is evident
that $\overline{56+x} \times 18$, or $1008+18x$, is the value of the whole mixture. And therefore
$1120+14x=1008+18x$.
Whence $112=4x$.
And conſequently $28=x$.

## QUESTION XXV.

*A Farmer would mix Wheat, at 4 Shillings a Buſhel, with Rye, at 2s. 6d. a Buſhel; ſo that the whole Mixture may conſiſt of 90 Buſhels, and be afforded at 3s. 2d. a Buſhel. It is required to find how many Buſhels of each Sort muſt be taken.*

If the number of buſhels of wheat be $x$,
Thoſe of rye will be $90-x$.
Moreover, the value of the wheat will be $48x$, pence, and the value of the rye $\overline{90-x} \times 30$, pence.
Whence $48x+\overline{90-x} \times 30=90 \times 38$ *(by the queſtion.)*
that is, $48x+2700-30x=3420$:
therefore $18x=720$:
and conſequently $x=\frac{720}{18}=40$.

## QUESTION XXVI.

*A Workman was hired for 40 Days, at 3s. 4d. per Day, for every Day he worked; but with this Condition, that for every Day he played, he was to forfeit 1s. 4d: and it ſo happened, that, upon the whole, he had 3l. 3s. 4d. to receive. The queſtion is, to find how many Days of the 40 he worked, and how many he played.*

Let $x$ be the number of days he worked;
then $40-x$ will be the number of days he played.
Moreover, as he was to receive 40 pence, for every day he worked, and to forfeit 16 pence for every day he played,



Digitized by Google

we have $40x$ = the number of pence earned by work,
and $\overline{40-x} \times 16$ = the number of pence forfeited by play:
whence $40x - \overline{40-x} \times 16 = 760$, *by the queſtion*;
that is, $40x - 640 + 16x = 760$:
therefore $56x = 1400$;
and conſequently $x = 25$.
From which it is evident that he worked 25 days, and played 15.

## QUESTION XXVII.

*A Bill of* 120*l. was paid in Guineas and Moidores, and the Number of Pieces uſed of both Sorts was juſt* 100. *How many were there of each?*

If $x$ be put for the number of guineas;
then $100-x$ will be the number of moidores.
And ſo, $21x + \overline{100-x} \times 27 = 120 \times 20$
or, $21x + 2700 - 27x = 2400$. } *by the queſtion.*
Hence $300 = 27x - 21x = 6x$.
And conſequently $x = \frac{300}{6} = 50$.

## QUESTION XXVIII.

*One bought* 30 *Pounds of Sugar, of two different Sorts, and paid for the whole* 19 *Shillings; the beſt Sort coſt* 10*d* per *Pound, and the worſt* 7*d. How many Pounds were there of each?*

Let $x$ ſtand for the number of pounds of the beſt ſort,
and then $30-x$ will expreſs the pounds of the other ſort.
Therefore $x \times 10 + \overline{30-x} \times 7 = 19 \times 12$, *by the queſtion*;
that is, $10x + 210 - 7x = 228$.
Whence $3x = 228 - 210 = 18$,
and $x = 6$. Therefore there were 6 pounds of the beſt ſort, and 24 of the worſt.

QUESTION

Digitized by Google

## QUESTION XXIX.

*A Lady gave a Guinea in Charity, among a Number of Poor, conſiſting of Men, Women, and Children: each Man had 12d. each Woman 6d. and each Child 3d. Moreover there were twice as many Women as Men, wanting 2; and 3 times as many Children as Women, wanting 4. How many Perſons were there relieved?*

Let $x$ be the number of the men;
then $2x-2$, will be the number of women,
and $6x-10$, the number of children.

Hence $x \times 12 + \overline{2x-2} \times 6 + \overline{6x-10} \times 3 = 21 \times 12$;
that is, $12x+12x-12+18x-30=252$,
or, $42x=294$.
Whence $x=7$.
Therefore there were 7 men, 12 women, and 32 children; in all 51 perſons.

## QUESTION XXX.

*To find that Number, which being divided, either into three or four equal Parts, the continual Product of all the Parts, in both Caſes, ſhall be exactly the ſame.*

Let $x$ be the required number;
ſo ſhall the continual product of the three equal parts be $\frac{x}{3} \times \frac{x}{3} \times \frac{x}{3} = \frac{x^3}{27}$; and that of the four equal parts $\frac{x}{4} \times \frac{x}{4} \times \frac{x}{4} \times \frac{x}{4} = \frac{x^4}{256}$.
Whence $\frac{x^4}{256} = \frac{x^3}{27}$, *by the queſtion.*
Therefore $27x=256$; and $x=9\frac{13}{27}$.

QUESTION

Digitized by Google

## QUESTION XXXI.

*To find two Numbers, in the Proportion of* 3 *to* 4, *whose Sum is to the Sum of their Squares, as* 7 *to* 50.

Let $3x$ denote the leſſer number:
then $4x$ will expreſs the greater.
And we ſhall have $3x+4x : 9x^2+16x^2 :: 7 : 50$,
or, $7x : 25x^2 :: 7 : 50$, *by the queſtion.*
Therefore $25x^2 \times 7 = 7x \times 50$,
or, $25x^2 = 50x$;
whence $x = \frac{50}{25} = 2$. So that 6 and 8 are the two numbers that anſwer the queſtion.

## QUESTION XXXII.

*To find two Numbers in the Proportion of* 9 *to* 7; *ſo that the Square of their Sum, and the Cube of their Difference, ſhall be equal.*

If $9x$ be put for the greater number;
then $7x$ will be the leſſer;
And ſo $\overline{16x}^2 = \overline{2x}^3$, *by the queſtion,*
that is, $256x^2 = 8x^3$.
Hence $x = \frac{256}{8} = 32$.
Therefore 288 and 224, are the two numbers ſought.

## QUESTION XXXIII.

*To find two Numbers whoſe Difference is* 4 *and the Difference of their Squares* 120.

Let $x$ be the leſſer number;
then $x+4$, will be the greater:
alſo $xx$ will be the ſquare of the leſſer,
and $xx+8x+16$ that of the greater.
Whence $8x+16=120$, *by the queſtion.*
Therefore $8x=104$: and $x=13$.
So that 13 and 17 are the two numbers that were to be found.

QUESTION

Digitized by Google

## QUESTION XXXIV.

*To divide* 100 *into two ſuch Parts, that the Difference of their Squares may be* 1000.

If $x$ be the greater part,
$100-x$, will be the leſſer.
Therefore $xx-\overline{100-x}|^2=1000$;
that is, $x^2-10000+200x-x^2=1000$.
Whence $200x=11000$;
and conſequently $x=\frac{11000}{200}=55$.

## QUESTION XXXV.

*To divide* 100 *into two Parts, ſo that the Square of their Difference may exceed the Square of twice the leſſer Part by* 2000.

The leſſer part being denoted by $x$,
the greater will be expreſſed by $100-x$,
and the difference by $100-2x$.
Therefore, by the problem, $\overline{100-2x}|^2=\overline{2x}|^2+2000$,
that is, $10000-400x+4xx=4xx+2000$,
or, $10000-2000=400x$.
Hence $x=\frac{8000}{400}=20$; and $100-x=80$.

## QUESTION XXXVI.

A *and* B *make a joint Stock of* 500*l. by which they gain* 160*l. whereof* A, *for his Share, had* 32*l. more than* B. *What did each Perſon bring into Stock?*

If $x$ be the number of pounds advanced by A;
then it will be, as 500 (the whole ſtock) is to 160 (the whole gain) ſo is $x$ (the ſtock of A) to $\frac{160x}{500}$, the gain of A: whence the gain of B being $160-\frac{16x}{50}$:

we

Digitized by Google

we have $\frac{16x}{50}=160-\frac{16x}{50}+32$, *by the queſtion.*

Therefore $\frac{32x}{50}=192$, or $\frac{x}{50}=6$;

and conſequently $x=50\times 6=300$.

## QUESTION XXXVII.

*A Sum of Money was divided between two Perſons,* A *and* B, *ſo that the Share of* A *was to that of* B, *as* 5 *to* 3, *and exceeded* $\frac{1}{8}$ *of the whole Sum by* 50*l*. *What was the Share of each Perſon?*

Let $5x$ expreſs the ſhare of A,

and $3x$ the ſhare of B;

then $8x$ will be the whole ſum,

and $\frac{40x}{9}$ will be $\frac{5}{9}$ thereof.

Therefore $5x-\frac{40x}{9}=50$, *by the queſtion.*

that is, $\frac{5x}{9}=50$, or $x=90$.

Hence 450*l*. and 270*l*. are the two ſhares required.

## QUESTION XXXVIII.

A *and* B *began to play together with equal Sums of Money*; A *firſt won* 20 *Guineas, but afterwards loſt back the Half of all he then had; and thereupon had only half as much Money as* B. *What Money did each begin with?*

Let $x$ be the number of guineas required:

then A, after winning 20 guineas, had $x+20$; the half of which, or $\frac{x}{2}+10$, is therefore what he had at laſt: and this deducted from $2x$ (the whole ſum betwixt both) leaves $2x-\frac{x}{2}-10=$ what B had at laſt.

Therefore

Digitized by Google

Therefore $2x-\frac{x}{2}-10=x+20$;

whence $3x-20=2x+40$,
and consequently $x=60$.

## QUESTION XXXIX.

*A Gentleman left his whole Eſtate among his four Sons; whereof the Eldeſt had $\frac{1}{2}$ wanting 800l. the Second $\frac{1}{4}$ and 120l. over; the Third had half as much as the Eldeſt; and the Youngeſt $\frac{2}{3}$ of what the Second had. What was the whole Eſtate? and how much had each?*

Let $x$ be the whole eſtate; then

The firſt had $\frac{x}{2}-800$.

The Second $\frac{x}{4}+120$.

The third $\frac{x}{4}-400$.

The fourth $\frac{2x}{12}+80$.

The ſum of all which is, conſequently, equal to the whole,

or, $\frac{x}{2}+\frac{x}{4}+\frac{x}{4}+\frac{x}{6}-1000=x$.

But it is plain (without reduction)

that $\frac{x}{2}+\frac{x}{4}+\frac{x}{4}=x$.

Hence our equation becomes

$\frac{x}{6}-1000=0$, or $\frac{x}{6}=1000$.

Therefore the whole eſtate was 6000l. whereof the eldeſt ſon had 2200l. the ſecond 1620l. the third 1100l. and the youngeſt 1080l.



Digitized by Google

## QUESTION XL.

*One being asked his Age, replied; If $\frac{2}{5}$ of my Years be multiplied by 3, and $\frac{1}{3}$ of them be added to the Product, the Amount will be 115. What was his Age?*

If $x$ be the required number of years,
then $\frac{2x}{5} \times 3 + \frac{x}{3} = 115$, *by the queſtion*;
that is, $\frac{6x}{5} + \frac{x}{3} = 115$.
Therefore $18x + 5x = 15 \times 115$,
or $23x = 15 \times 115$.
Conſequeutly $x = \frac{15 \times 115}{23} = 15 \times 5 = 75$.

## QUESTION XLI.

*A Perſon being aſked the Hour of the Day, anſwered thus: If $\frac{3}{4}$ of the Number of Hours remaining till Midnight be multiplyed by 4, the Product will as much exceed 12 Hours, as Half the preſent Hour from Noon is below 4. What was the Hour after Noon?*

Let $x$ be the required hour;
then $12 - x$ will be the hours till midnight,
and $\frac{36 - 3x}{8} \times 4 - 12 = 4 - \frac{x}{2}$, *by the queſtion.*
That is, $\frac{36 - 3x}{2} - 12 = 4 - \frac{x}{2}$,
whence $36 - 3x - 24 = 8 - x$;
and conſequently $x = 2$.

QUESTION

Digitized by Google

## QUESTION XLII.

*A Market-woman bought in a certain Number of Eggs, at the Rate of 5 for two Pence; one half of which she sold out again at 2 a Penny, and the remaining Half at 3 a Penny; and cleared 4 Pence by so doing. What Number of Eggs had she?*

Let $2x$ be the number sought;
then, *by the question.*

$5 : 2 :: 2x : \frac{4x}{5}$ the number of pence the eggs cost.

But the number of pence they were sold for, again, is $\frac{x}{2}+\frac{x}{3}$: thereforefore we have this equation,

*viz.* $\frac{x}{2}+\frac{x}{3}-\frac{4x}{5}=4$.

From whence $15x+10x-24x=120$, or $x=120$.

## QUESTION XLIII.

*A certain Sum of Money, put out at Interest, amounts, in 8 Months, to 297l. 12s. And, in 15 Months its Amount (computed according to simple Interest) is 306l. What is that Sum? And what the Rate of Interest?*

Let $x$ be the number of pounds in the required sum:
then, the interest thereof for 8 months being $297,6-x$,
and for 15 months $306-x$, we have,
as $8 : 15 :: 297,6-x : 306-x$.
Whence, by multiplying extremes and means,
we get $2448-8x=4464-15x$.
Therefore $7x=2016$,
and consequently $x=288l.$ the sum required.
For the rate of interest, it will be,
as $288l. \times 15 : 100l. \times 12 :: 18l.$ (the interest of 288*l.* for 15 months) to 5*l.* the required interest of 100*l.* for 12 months.



Digitized by Google

## QUESTION XLIV.

*A Waterman finds by Experience, that he can, with the Advantage of a common Tide, row from London to Greenwich, which is 5 Miles, in 3 Quarters of an Hour; and that, to return to London, againſt an equal Tide, though he rows back along ſhore, where the Stream is only half as ſtrong as in the Middle, takes him a full Hour and Half. 'Tis required to find, from hence, at what rate* per *Hour, the tide runs in the Middle where it is ſtrongeſt.*

In the firſt place it will be

qr. qr. m. m.
3 : 4 : : 5 : $6\frac{2}{3}$ = diſt. rowed *per* hour with the tide,
6 : 4 : : 5 : $3\frac{1}{3}$ = diſt. rowed *per* hour againſt tide.

If now the former of theſe two diſtances ($6\frac{2}{3}$) be put $= a$, and the latter ($3\frac{1}{3}$) $= b$; and $x$ be aſſumed to expreſs the required diſtance run, *per* hour, by the ſtream in the middle of the river; then $a - x$ will be the *real* effect of his rowing, *per* hour, in going from London, the motion of the tide being deducted; and $b + \frac{x}{2}$ will be the like effect in his return:

and ſo, theſe two quantities being equal to each other, we have $b + \frac{x}{2} = a - x$.

Whence $2b + x = 2a - 2x$;

and conſequently $x = \frac{2a - 2b}{3} = 2\frac{2}{3}$ m.

## QUESTION XLV.

*To divide* 36 (a) *into* 3 *ſuch Parts, that* $\frac{1}{2}$ *of the Firſt,* $\frac{1}{3}$ *of the Second, and* $\frac{1}{4}$ *of the Third, may be equal to each other.*

If $x$ be put for the firſt part,

then is $\frac{x}{2} = \frac{1}{3}$ of the ſecond part *(by the queſtion).*

And

Digitized by Google

And so $\frac{3x}{2}$ = second part.

Moreover $\frac{x}{2}$ being $=\frac{1}{4}$ of the third part,

therefore $\frac{4x}{2}$ (or $2x$) = third part.

Hence $x+\frac{3x}{2}+2x=a$,

and $2x+3x+4x=2a$.

Consequently $x=\frac{2a}{9}=\frac{2\times 36}{9}=2\times 4=8$,

From which the second part $\left(\frac{3x}{2}\right)$

appears to be $=12$, and the third $(2x)$ $=16$.

## QUESTION XLVI.

*To divide the Number* 90 *into* 4 *such Parts, that, if the first be increased by* 5, *the second diminished by* 4, *the third multiplied by* 3, *and the fourth divided by* 2, *the Result, in each Case, shall be exactly the same.*

Let $x$ be the fourth, or last, part:

then, three times the third part being $=\frac{x}{2}$,

the third part will be $\frac{x}{6}$.

Moreover, the second part $-4$ being, also, $=\frac{x}{2}$,

the second part will be $\frac{x}{2}+4$:

and, the first part $+5$ being $=\frac{x}{2}$,

the first part, alone, will be $\frac{x}{2}-5$.

And, by adding all the parts thus found together,

we have $x+\frac{x}{6}+\frac{x}{2}+4+\frac{x}{2}-5=90$;

that

Digitized by Google

that is, $2x+\frac{x}{6}-1=90$.

Whence $13x=91\times 6$: and $x=42$.

Therefore the four required parts are 16, 25, 7, and 4 2/3, respectively.

## QUESTION XLVII.

*Two Workmen,* A *and* B, *were employed together for* 50 *Days, at* 5 *Shillings* per *Day, each; during which Time* A, *by ſpending only Sixpence a Day leſs than* B, *had ſaved twice as much as* B, *beſides the Expence of* 2 *Days over. What did each Perſon expend a Day?*

Let $x$ be the pence A ſpent *per* day;
then $60-x$, will be what he ſaved *per* day,
and $54-x$, what B ſaved.
Therefore $3000-50x$ are A's whole ſavings,
and $2700-50x$ thoſe of B.

Hence $3000-50x=2\times 2700-50x+2x$;
or, $3000-50x=5400-98x$.
From which $48x=2400$, and $x=50$.

## QUESTION XLVIII.

*Two Perſons,* A *and* B, *have both the ſame Income; A lays by* $\frac{1}{5}$ *of his; but* B, *by ſpending* 60*l.* per Ann. *more than* A, *at the End of three Years finds himſelf* 100*l. in debt. What did each receive, and expend,* per Annum?

Let $x$ be the yearly income of each;

then $\frac{4x}{5}$ is the ſum expended by A, *per ann.*

and $\frac{4x}{5}+60$, that expended by B.

Therefore $\frac{4x}{5}+60-x$, is what B runs in debt.

Conſequently $\frac{4x}{5}+60-x\times 3=100$,

or

or $\frac{12x}{5}+180-3x=100$;

that is, $180-\frac{3x}{5}=100$.

Whence $900-3x=500$,

and $x=\frac{400}{3}=133l.\ 6s.\ 8d.$

Therefore A expended 106*l.* 13*s.* 4*d.* and B 166*l.* 13*s.* 4*d.* *per annum.*

## QUESTION XLIX.

*A Grazier bought in as many Sheep, of different Sorts, as coſt him* 33*l.* 7*s.* 6*d. For the firſt Sort, which were* $\frac{1}{3}$ *of the whole, he paid* 9*s.* 6*d. a-piece; for the ſecond Sort, which were* $\frac{1}{4}$ *of the whole, he paid* 11*s. each; and for the reſt,* 12*s.* 6*d. each. What Number of Sheep did he buy in all?*

If $x$ be the whole number of ſheep;

then, the number of the firſt ſort being $\frac{x}{3}$, and of the ſecond ſort $\frac{x}{4}$, the number of the remaining ſort (at 12*s.* 6*d.* each) muſt be $x-\frac{x}{3}-\frac{x}{4}=\frac{12x-4x-3x}{12}=\frac{5x}{12}$:

whence, by the conditions of the problem, we have

$\frac{x}{3}\times 19+\frac{x}{4}\times 22+\frac{5x}{12}\times 25=1335$;

that is, $\frac{19x}{3}+\frac{22x}{4}+\frac{125x}{12}=1335$.

Let each term of this equation be now multiplied by 12, and it will become

$76x+66x+125x=16020$,

or, $267x=16020$.

Therefore $x=60$.

QUESTION

Digitized by Google

## QUESTION L.

*A Draper, of a Piece of Cloth, ſtanding him in* 3*s.* 2*d.* per *Yard, ſold* $\frac{1}{3}$ *Part, at* 4*s.* per *Yard*; $\frac{1}{4}$ *at* 3*s.* 8*d.* per *Yard*; $\frac{1}{5}$ *at* 3*s.* 6*d.* per *Yard*; *and the Remnant at* 3*s.* 4*d. a Yard: and his Gain upon the Whole, was* 15*s.* 2*d. How many Yards did the Piece contain?*

If the number ſought be denoted by $x$;
then the number of yards in the remnant
will be $x-\frac{x}{3}-\frac{x}{4}-\frac{x}{5}=\frac{60x-20x-15x-12x}{60}=\frac{13x}{60}$.
Therefore by the queſtion, we have
$\frac{x}{3}\times 48+\frac{x}{4}\times 44+\frac{x}{5}\times 42+\frac{13x}{60}\times 40-38x=182$,
or, $16x+11x+\frac{42x}{5}+\frac{26x}{3}-38x=182$,
that is $\frac{42x}{5}+\frac{26x}{3}-11x=182$:
whence $126x+130x-165x=2730$.
And therefore $x=\frac{2730}{91}=30$.

## QUESTION LI.

*A Diſtiller propoſes to mix Foreign Brandy, ſtanding him in* 8 *Shillings a Gallon, with Britiſh Spirits of* 3 *Shillings* per *Gallon, in ſuch Proportion that he may gain* 30 per Cent *by ſelling out the Compound at* 9*s. a Gallon. What is that Proportion?*

Suppoſe, that, with $a$ Gallons of Brandy, he mixes $x$ gallons of ſpirits; then, the brandy, ſtanding him in $8a$ (ſhillings) and the ſpirits in $3x$ (ſhillings), the true value of the whole mixture will be $8a+3x$.
But the value of $a+x$ gallons, at 9 ſhillings *per* gallon, is $9a+9x$: therefore, by laying out $8a+3x$, he gains $a+6x$: and ſo
we have $8a+3x : a+6x :: 100 : 30$, *by the queſtion.*
Conſequently

Digitized by Google

Conſequently $100a+600x=240a+90x$;
whence $510x=140a$,
and $x=\frac{14a}{51}$.

From which it appears, that to every 51 gallons of brandy, there muſt be taken 14 gallons of ſpirits.

## QUESTION LII.

*To find two Numbers in the Proportion of* 4 *to* 5, *from which two other (required) Numbers, in the Proportion of* 6 *to* 7 *being, reſpectively, deducted, the Remainders ſhall be in the Proportion of* 2 *to* 3, *and their Sum equal to* 20.

Let $4x$ and $5x$ be the two firſt numbers,
and $6y$ and $7y$ the other 2 numbers.
Then $4x-6y : 5x-7y :: 2 : 3$
And $9x-13y=20$ } *by the queſtion.*
From which proportion, by multiplying extremes and means, we have $12x-18y=10x-14y$, and
therefore $x=2y$; which ſubſtituted in the above equation gives $18y-13y=20$;
whence $y=\frac{20}{5}=4$; and $x\ (=2y)=8$.

Therefore the 2 firſt numbers are 32 and 40; and the other two, 24 and 28.

## QUESTION LIII.

*A Farmer ſold, at one time,* 30 *Buſhels of Wheat and* 40 *of Barley, and for the whole received* 13*l.* 10*s*; *and, at another time, he ſold* 50 *Buſhels of Wheat and* 30 *of Barley, at the ſame Prices as before, and for the whole received* 17*l. The Queſtion is, to find what each Sort of Grain was ſold at* per *Buſhel*

Let $x$ and $y$ expreſs the numbers of ſhillings, reſpectively, that the wheat and barley were ſold at *per* buſhel; and then, from the conditions of the queſtion, we ſhall have the two following equations, *viz.*



Digitized by Google

$30x+40y=270$,
$50x+30y=340$.

From 4 times the second of which equations let 3 times the first be subtracted, and there will remain $110x=550$.

Therefore $x=\frac{55}{11}=5$:

and $y\left(=\frac{27-3x}{4}\right)=3$.

## QUESTION LIV.

*A Farmer, with 28 Bushels of Barley, at 2s. 4d. per Bushel, would mix Rye, at 3s. per Bushel, and Wheat, at 4s. per Bushel; so that the whole Mixture may consist of 100 Bushels, and be worth 3s. 4d. a Bushel. How many Bushels of Rye, and how many of Wheat must he mingle with the Barley?*

Let $x$ be the number of bushels of rye, and $y$ those of the wheat: then, the value of the barley being 784 (pence), of the rye $36x$ (pence), and of the wheat $48y$ (pence), we have

$$\left.\begin{array}{l}784+36x+48y=4000\\ \text{and } 28+x+y=100\end{array}\right\}\ \textit{by the question.}$$

From the first of which equations, take 36 times the second, and there results,

$784-36\times 28+12y=400$,

that is, $-224+12y=400$.

Therefore $12y=624$,

and consequently $y=\frac{624}{12}=52$.

Whence $x\ (=100-28-y)=20$.

QUESTION

Digitized by Google

## QUESTION LV.

A *and* B, *working together on the ſame Work, can earn* 40 *Shillings in* 6 *Days*; A *and* C *together can earn* 54 *Shillings in* 9 *Days*; *and* B *and* C, 80 *Shillings in* 15 *Days. It is required to find what each Perſon, alone, can earn* per *Day.*

Let $x$, $y$ and $z$ expreſs the numbers of ſhillings in the three required values, reſpectively.

Then $\left\{\begin{array}{l} 6x+6y=40 \\ 9x+9z=54 \\ 15y+15z=80 \end{array}\right\}$ *by the queſtion.*

And $\left\{\begin{array}{l} x+y=6\frac{2}{3} \\ x+z=6 \\ y+z=5\frac{1}{3} \end{array}\right\}$ *by diviſion.*

Hence $y-z=\frac{2}{3}$, *by ſubtracting the* 2*d equation from* 1*ſt.*
And $2y=6$, *by adding the two laſt.*
Conſequently $y=3$.
From which we have $x\ (=6\frac{2}{3}-y)=3\frac{2}{3}=3s.\ 8d.$ and $z\ (=5\frac{1}{3}-y)=2\frac{1}{3}=2s.\ 4d.$

## QUESTION LVI.

*Fo find three Numbers, ſo that* $\frac{1}{2}$ *the firſt,* $\frac{1}{3}$ *of the ſecond, and* $\frac{1}{4}$ *of the third, ſhall together be equal to* 62; *alſo* $\frac{1}{3}$ *of the firſt,* $\frac{1}{4}$ *of the ſecond, and* $\frac{1}{5}$ *of the third, equal to* 47; *and, laſtly,* $\frac{1}{4}$ *of the firſt,* $\frac{1}{5}$ *of the ſecond, and* $\frac{1}{6}$ *of the third, equal to* 38.

Put $a=62$, $b=47$, and $c=38$; and let the three required numbers be denoted by $x$, $y$ and $z$, reſpectively; then the conditions of the problem will be expreſſed in the three following equations, *viz.*

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=a.$$
$$\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=b,$$
$$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=c.$$



Digitized by Google

Which, cleared of fractions, become

$12x+8y+6z=24a$

$20x+15y+12z=60b$

$30x+24y+20z-120c.$

Now (in order to exterminate $z$) let the second of these equations be taken from the double of the first, and also the treble of the third from the quintuple of the second; and there results

$4x+y=48a-60b$; and

$10x+3y=300b-360c$;

whence, by deducting the second of these from the treble of the former, and dividing by 2, there comes out

$x=72a-240b+180c=24.$

From which $y$ $(=48a-60b-4x)$ is also found $=60$, and $z$ $(=a-\frac{1}{2}x-\frac{1}{3}y\times 4)$ $=120$.

## QUESTION LVII.

*To divide the Number* 90 (a) *into three Parts, so that, the Double of the first Part* +40 (*b*); *the Treble of the second* +20 (c); *and the Quadruple of the third* +10 (d), *may be all equal to one another.*

Let $x$, $y$, and $z$ represent the three required parts, respectively; then, from the conditions of the problem, we shall have

$x+y+z=a,$

$2x+b=3y+c,$

$2x+b=4z+d.$

Now, in order to exterminate $y$ and $z$, let 12 times the first equation, 4 times the second, and 3 times the third, be added all together; and you will have

$26x+12y+12z+7b=12a+12y+12z+4c+3d.$

Therefore $26x=12a+4c+3d-7b,$

and $x=\frac{12a+4c+3d-7b}{26}=35.$

Whence $y\left(=\frac{2x+b-c}{3}\right)=30$, and $z$ $(=a-x-y)$ $=25$.

QUESTION

## QUESTION LVIII.

*To find three Numbers, ſo that the firſt with half the other two, the ſecond with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, may be the ſame, and amount to* 51 *in each Caſe.*

Put $a=51$, and let $x$, $y$ and $z$ denote the three required numbers; then, by the queſtion,

$$x+\frac{y+z}{2}=a,$$

$$y+\frac{x+z}{3}=a,$$

$$z+\frac{x+y}{4}=a.$$

Which, cleared of fractions, become

$$2x+y+z=2a,$$
$$x+3y+z=3a,$$
$$x+y+4z=4a.$$

From whence, by taking the ſecond from the third, and the firſt from the double of the ſecond, there reſults $-2y+3z=a$,
and $5y+z=4a$.
And, by deducting the former of theſe, from the treble of the latter, we have $17y=11a$,

Therefore $y=\frac{11a}{17}=33$,

$z\ (=4a-5y)\ =\frac{13a}{17}=39$,

and $x\ (=3a-3y-z)\ =\frac{5a}{17}=15$,

QUESTION

Digitized by Google

## QUESTION LIX.

*A certain Sum of Money was divided between three Perſons,* A, B *and* C; *ſo that* A's *Share exceeded* $\frac{4}{7}$ *of the Shares of* B *and* C *by* 30*l*; *alſo the Share of* B *exceeded* $\frac{3}{8}$ *of the Shares of* A *and* C *by* 30*l*; *and the Share of* C *likewiſe exceeded* $\frac{2}{9}$ *of the Shares of* A *and* B, *by* 30*l*. *The Queſtion is, to find the Share of each Perſon.*

Let $a=30$; and let $x$, $y$, and $z$ be aſſumed to expreſs the three required numbers; then by the conditions of the problem,

$$x-\frac{4y+4z}{7}=a,$$

$$y-\frac{3x+3z}{8}=a,$$

$$z-\frac{2x+2y}{9}=a.$$

Whence, by reduction,

$$7x-4y-4z=7a,$$
$$-3x+8y-3z=8a,$$
$$-2x-2y+9z=9a.$$

Now, to get rid of $y$ (which, becauſe of the even coefficents, is eaſieſt to be exterminated) let the double of the firſt equation and the quadruple of the third be, ſucceſſively, added to the ſecond; by means whereof we have

$$11x-11z=22a,$$
$$-11x+33z=44a.$$

Moreover, by adding theſe two laſt equations together, we have $22z=66a$,

Therefore $z=3a=90$;

whence $x\ (=2a+z)\ =5a=150$,

and $y\ \left(=a+\frac{3}{8}\times\overline{x+z}\right)\ =4a=120$.

QUESTION

## QUESTION LX.

*If* A *and* B *together can perform a Piece of Work in* 8 *Days*; A *and* C *together in* 9 *Days*; *and* B *and* C *in* 10 *Days. How many Days will it take each Perſon, alone, to perform the ſame Work?*

Let the whole work be repreſented by $a$, and let $x$, $y$, and $z$ ſtand for the parts thereof performed by A, B, and C in one day, reſpectively.

then will $\left\{\begin{matrix} 8x+8y=a \\ 9x+9z=a \\ 10y+10x=a \end{matrix}\right\}$ *by the queſtion.*

And $\left\{\begin{matrix} x+y=\frac{a}{8} \\ x+z=\frac{a}{9} \\ y+z=\frac{a}{10} \end{matrix}\right\}$ *by diviſion.*

Whence $y-z\ \left(=\frac{a}{8}-\frac{a}{9}\right)=\frac{a}{72}$,

and $2y=\frac{a}{10}+\frac{a}{72}=\frac{82a}{720}$.

Conſequently $y=\frac{41a}{720}$: from which $x\ \left(=\frac{a}{8}-y\right)$ is found $=\frac{49a}{720}$,

and $z\ \left(=\frac{a}{10}-y\right)=\frac{31a}{720}$.

Now, the part of the work ($a$) performed by each perſon in one ſingle day being thus aſſigned, the number of days it will take any one of them to do the whole, will be found by dividing the whole by the aſſigned part.

Thus, $\left(\frac{41a}{720}\right)\ a=\frac{720a}{41a}=\frac{720}{41}=17\frac{23}{41}$ is the number of days in which B, alone, can do the whole. And, in like

like manner, the number of days in which A, or C, can do the whole, appears to be

$14\frac{34}{49}$, or $23\frac{7}{31}$, respectively.

## QUESTION LXI.

*If* A, B *and* C *can, together, finish a Piece of Work in* 9 *Days*; A, B *and* D *together, in* 10 *Days*; A, C *and* D *together, in* 11 *Days*; *and* B, C *and* D *in* 12 *Days*. *In how long Time can they all four, together, finish it?*

Here, denoting the given numbers by $a$, $b$, $c$, and $d$, and putting $u$, $x$, $y$, and $z$, for the parts of the whole work ($g$) done by each in one day, respectively, we shall, by the question, have these equations,

$$\text{Viz.} \left\{\begin{array}{l} a \times \overline{u+x+y} = g \\ b \times \overline{u+x+z} = g \\ c \times \overline{u+y+z} = g \\ d \times \overline{x+y+z} = g \end{array}\right\}$$

$$\text{or} \left\{\begin{array}{l} u+x+y = \frac{g}{a} \\ u+x+z = \frac{g}{b} \\ u+y+z = \frac{g}{c} \\ x+y+z = \frac{g}{d} \end{array}\right\} \textit{by division.}$$

The sum of all which, divided by 3,

gives $u+x+y+z = \frac{1}{3} \times \overline{\frac{g}{a}+\frac{g}{b}+\frac{g}{c}+\frac{g}{d}}$,

Therefore seeing the work done by all the four, in one day, is expressed by $\frac{1}{3} \times \overline{\frac{g}{a}+\frac{g}{b}+\frac{g}{c}+\frac{g}{d}}$, the whole work $g$, divided hereby, will consequently give

give $\frac{3g}{\frac{g}{a}+\frac{g}{b}+\frac{g}{c}+\frac{g}{d}} = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}} =$
$\frac{3abcd}{bcd+acd+abd+abc} = 7\frac{1797}{2289}$, for the number of days required.

## QUESTION LXII.

*To find that Number whose Square Root is to its Cube Root, in the Proportion of* 5 *to* 2.

Let $x^6$ expreſs the required number:
then $x^3$ will be its ſquare root,
and $x^2$ its cube root.
Now $x^3 : x^2 :: 5 : 2$, *by the queſtion.*
Therefore $2x^3 = 5x^2$, or $2x = 5$;
or, laſtly, $x = \frac{5}{2} = 2.5$.
Whence $x^6 = 244.140625 =$ the number ſought.

## QUESTION LXIII.

*To find two Numbers in the Proportion of* 3 *to* 5; *whereof the fifth Power of the firſt ſhall be to the third Power of the ſecond, as* 972 *to* 125.

If $3x$ be put for the firſt number,
then $5x$ will expreſs the ſecond; and we ſhall
have $\overline{3x}^5 : \overline{5x}^3 :: 972 : 125$, *by the queſtion,*
that is, $243x^5 : 125x^3 :: 972 : 125$.
Hence $243x^5 \times 125 = 125x^3 \times 972$,
or, $243x^2 = 972$.
Therefore $x^2 = \frac{972}{243} = 4$;
and $x = \sqrt{4} = 2$. So that 6 and 10 are the two numbers that were to be found.



Digitized by Google

## QUESTION LXIV.

*To find three Numbers in the Ratio of* $\frac{1}{2}$, $\frac{1}{3}$, *and* $\frac{1}{4}$; *whereof the Sum of all the Squares ſhall be* 549.

Let $x$ denote the firſt number,

then it will be $\frac{1}{2} : \frac{1}{3} :: x : \frac{2x}{3}$ = the ſecond number.

And $\frac{1}{2} : \frac{1}{4} :: x : \frac{x}{2}$ = the third number.

Hence $x^2+\overline{\frac{2x}{3}}^2+\overline{\frac{x}{2}}^2=549$, *by the queſtion.*

that is, $x^2+\frac{4x^2}{9}+\frac{x^2}{4}=549$,

or, $36x^2+16x^2+9x^2=36\times 549$.

From which $x^2=\frac{36\times 549}{61}=36\times 9$,

and $x=6\times 3=18$. Therefore 18, 12 and 9, are the three required numbers.

*Otherwiſe.*

By reducing the given fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, to the ſame denomination, they will appear to be in the proportion of 6, 4 and 3. If, therefore, the firſt of the three numbers ſought be denoted by $6x$, the other two will be expreſſed by $4x$, and $3x$, reſpectively: and ſo we ſhall have

$36x^2+16x^2+9x^2=549$.

Whence $x=3$, and the numbers ſought, *as before.*

## QUESTION LXV.

*Having given the Difference of two Numbers* $=6$, *and their Product* $=720$; *to find the Numbers.*

Let the leſſer of them be denoted by $x$; then the greater will be $x+6$; and ſo, by the queſtion, we ſhall have $xx+6x=720$.

But in order to the reſolution of this equation (in which both the firſt and ſecond powers of $x$ are involved)

Digitized by Google

volved) let half the coefficient of $x$, which (in this caſe) is 3, be taken and ſquared, and let that ſquare be added to both ſides of the equation: by which means it becomes $xx+6x+9=729$.
Whereof the former part being now a compleat ſquare, its root, may, therefore, be extracted; and will be expreſſed by the ſaid half coefficient joined to $x$ with its proper ſign; that is, by $x+3$ (as may be very eaſily proved by the multiplication of $x+3$ into itſelf; whence $xx+6x+9$, the very quantity above is produced.) Hence it is evident, that $x+3=\sqrt{729}=27$ (for equal quantities have equal ſquare roots;) and conſequently $x=24$.

## QUESTION LXVI.

*The Sum of two Numbers being given* $=60$, *and the Sum of their Squares* $=1872$; *to find the Numbers.*

Let $x$ be the greater of them;
then $60-x$ will be the leſſer:
and therefore $x^2+\overline{60-x}|^2=1872$;
or $x^2+3600-120x+x^2=1872$:
whence $2x^2-120x=-1728$,
and $x^2-60x=-864$:
from which, by compleating the ſquare *(as in the laſt problem)* we get $x^2-60x+900$ $(=-864+900)$ $=36$:
and conſequently, by taking the root, $x-30=\sqrt{36}=6$.
Therefore $x=6+30=36$: and $60-x=24$: which are the two numbers that were to found.

But, to ſolve the problem in a more general manner (by letters) put the ſum of the two numbers $=a$, the ſum of their ſquares $=b$, and the greater number $=x$, *as before.*
Then will $x^2+a^2-2ax+x^2=b$, *by the queſtion.*
Hence $2x^2-2ax=b-a^2$,
and $x^2-ax=\frac{b}{2}-\frac{a^2}{2}$.



Digitized by Google

Where, half the coefficient of the second term being $\frac{a}{2}$, the compleated square will therefore be $x^2-ax+\frac{a^2}{4}\left(=\frac{b}{2}-\frac{a^2}{2}+\frac{a^2}{4}\right)=\frac{b}{2}-\frac{a^2}{4}$: from which, by extracting the root,

we have $x-\frac{a}{2}=\sqrt{\frac{b}{2}-\frac{aa}{4}}$;

and therefore $x=\sqrt{\frac{b}{2}-\frac{aa}{4}}+\frac{a}{2}$: which, if $a$ be taken $=60$, and $b=1872$, will come out $=36$, *the very same as before.*

## QUESTION LXVII.

*To divide the Number* 60 (a) *into two such Parts, that their Product may be* 864 (b).

If $x$ be put for the greater part, the lesser will be denoted by $a-x$; and we shall therefore have $ax-xx=b$; by the conditions of the question.
This equation, by changing the signs of all its terms (in order to have the highest power of $x$ affirmative) becomes $xx-ax=-b$.
Whence, by compleating the square,

we have $xx-ax+\frac{aa}{4}=-b+\frac{aa}{4}$;

and consequently $x-\frac{a}{2}=\sqrt{\frac{aa}{4}-b}$.

Therefore $x=\sqrt{\frac{aa}{4}-b}+\frac{a}{2}=36$, the greater part;

and $a-x=24$, the lesser.

QUESTION

Digitized by Google

## QUESTION LXVIII.

*To divide the Number* 60 (a) *into two Parts, ſo that the Square of the Greater multiplied by the Leſſer, added to the Square of the Leſſer multiplied by the Greater, may amount to* 51840 (b).

If $x$ be the greater part; then $a-x$ will be the leſſer;
and $x^2 \times \overline{a-x+a-x}^2 \times x = b$,
that is, $ax^2 - x^3 + a^2x - 2ax^2 + x^3 = b$,
or, $a^2x - ax^2 = b$.
Whence $x^2 - ax = -\frac{b}{a}$;
and, conſequently, $x = \sqrt{\frac{a^2}{4} - \frac{b}{a}} + \frac{a}{2} = 36$.

## QUESTION LXIX.

*The Sum of two Numbers being given* =20 (a), *and the Sum of their Cubes* =2240 (b); *to determine the Numbers,*

Let $x$ be the greater; then $a-x$ will be the leſſer;
and therefore $x^3 + \overline{a-x}^3 = b$,
that is, $x^3 + a^3 - 3a^2x + 3ax^2 - x^3 = b$:
whence $3ax^2 - 3a^2x = b - a^3$,
and $x^2 - ax = \frac{b}{3a} - \frac{a^2}{3}$.
Therefore, by compleating the ſquare,
$x^2 - ax + \frac{aa}{4}\left(= \frac{b}{3a} - \frac{a^2}{3} + \frac{a^2}{4}\right) = \frac{b}{3a} - \frac{a^2}{12}$:
and, by extracting the root, $x - \frac{a}{2} = \sqrt{\frac{b}{3a} - \frac{aa}{12}}$.
Conſequently $x = \frac{a}{2} + \sqrt{\frac{b}{3a} - \frac{aa}{12}} = \frac{20}{2} + \sqrt{37\frac{1}{3} - 33\frac{1}{3}}$
$= 12$.

QUESTION

## QUESTION LXX.

*To divide the Number* 240 (a) *into two ſuch Parts, that the greater Part divided by the Leſſer, may be to the leſſer Part divided by the Greater, in the Proportion of* 147 *to* 75 *(or of* m *to* n).

If the greater part be denoted by $x$, the leſſer will be expreſſed by $a-x$; and we ſhall have

$\frac{x}{a-x} : \frac{a-x}{x} :: m : n.$

Hence $\frac{nx}{a-x} = \frac{m \times \overline{a-x}}{x}$

and $\frac{nx^2}{m} = \overline{a-x}^2$:

from which, by extracting the ſquare root, on both ſides, $x\sqrt{\frac{n}{m}} = a-x.$

Whence, putting $\sqrt{\frac{n}{m}} = \frac{b}{c}$,

we have $bx = ca - cx$; and conſequently $x = \frac{ca}{b+c}$.

But, in the caſe above propoſed, $\sqrt{\frac{n}{m}}$ being $= \sqrt{\frac{75}{147}} = \sqrt{\frac{25}{49}} = \frac{5}{7}$, we have $b=5$, $c=7$; and therefore $x = \frac{7 \times 240}{12} = 7 \times 20 = 140$.

QUESTION

Digitized by Google

## QUESTION LXXI.

*Two Workmen,* A *and* B, *were employed, by the Day, at different Rates;* A *at the end of a certain Number of Days, had* 96 *Shillings to receive; but* B, *who played* 6 *of thoſe Days, received only* 54 *Shillings: but, had* B *worked the whole Time, and* A *played* 6 *Days, they would have received exactly alike. It is propoſed to find the Number of Days they were employd; and what each had a Day.*

Let $x$ be the number of days that A worked; then $x-6$ will be the days that B worked.

Moreover $\frac{96}{x}$, will be the wages of A *per* day;

and $\frac{54}{x-6}$, the wages of B *per* day.

Therefore $\frac{54}{x-6}\times x$, is what B would have earned, had he worked the whole time:

and $\frac{96}{x}\times\overline{x-6}$, what A would have earned had he played 6 days. Which two values being equal, by the queſtion, we have $\frac{54x}{x-6}=\frac{96\times\overline{x-6}}{x}$.

Whence, by reduction, $54x^2=96\times\overline{x-6}^2$,

or, $\frac{9x^2}{16}=\overline{x-6}^2$: therefore, by extracting the ſquare root on both ſides, $\frac{3x}{4}=x-6$; and conſequently $x=24$.

From which it is evident, that A had 4 ſhillings, and B 3 ſhillings, a day.

QUESTION

Digitized by Google

## QUESTION LXXII.

*From two Places, at the Diſtance of* 320 (a) *Miles, two Perſons,* A *and* B, *ſet out, at the ſame Time, in order to meet each other;* A *travelled* 8 (b) *Miles a Day more than* B; *and the Number of Days in which they met was equal to Half the Number of Miles* B *went in a Day. It is required to find how far each travelled to meet the other.*

Let $x$ be the number of days in which they met; then, by the queſtion, $\left\{\begin{matrix} 2x \\ 2x+b \end{matrix}\right\}$ will be the number of miles $\left\{\begin{matrix} \text{B} \\ \text{A} \end{matrix}\right\}$ went a day.

Therefore, by multiplying each of theſe by ($x$) the number of days, we have $2xx$, and $2xx+bx$, for the whole number of miles travelled by B and A, reſpectively:

and conſequently $4xx+bx=a$.

Hence $xx+\frac{bx}{4}+\frac{bb}{64}=\frac{a}{4}+\frac{bb}{64}$,

and $x=\sqrt{\frac{a}{4}+\frac{bb}{64}}-\frac{b}{8}=8$.

Therefore $2xx=128=$ the miles travelled by B; and $2xx+bx=192=$ thoſe travelled by A.

## QUESTION LXXIII.

*Two Meſſengers,* A *and* B, *were diſpatched at the ſame Time, to a Place, at the Diſtance of* 90 (a) *Miles; the former of whom, by riding one Mile an Hour more than the other, arrived at the end of his Journey one Hour before him. The Queſtion is, to find at what Rate each travelled* per *Hour.*

If $x$ be the miles that A rode *per* hour;

then $x-1$ will be the miles which B rode *per* hour:

moreover $\frac{a}{x}$ will be the number of hours in which A performed

Digitized by Google

performed the whole journey; and $\frac{a}{x-1}$ will be the number of hours wherein B performed it.

And therefore $\frac{a}{x}=\frac{a}{x-1}-1$, *by the queſtion.*

whence, by reduction $ax-a=ax-x^2+x$,

or $x^2-x=a$.

Therefore $x^2-x+\frac{1}{4}=a+\frac{1}{4}$ (by compleating the ſquare)

and conſequently $x=\sqrt{a+\frac{1}{4}}+\frac{1}{2}=9\frac{1}{2}+\frac{1}{2}=10$.

## QUESTION LXXIV.

*To find two Numbers, ſo that their Sum multiplied by the Greater, may produce* 100 *times the Leſſer, and being multiplied by the Leſſer may produce* 64 *times the Greater.*

Let $x$ denote the greater number, and $y$ the leſſer.

Then $\overline{x+y}\times x=100y$,

And $\overline{x+y}\times y=64x$.

Now, the firſt of theſe equations being multiplied by $y$, and the ſecond by $x$, they become both alike;

and ſo we have $100y^2=64x^2$:

therefore, by taking the ſquare root, $10y=8x$,

and conſequently $y=\frac{4x}{5}$: which value ſubſtituted in the ſecond equation, gives $\overline{x+\frac{4x}{5}}\times\frac{4x}{5}=64x$;

or, $\frac{9x}{5}\times\frac{4}{5}=64$.

Whence $x=\frac{25\times 16}{9}=44\frac{4}{9}$; and $y=35\frac{5}{9}$.



Digitized by Google

## QUESTION LXXV.

*To find three Numbers, ſo that their continual Product divided by the Sum of each two of them, may quote given Numbers; or (which is the ſame Thing) to determine the Values of $x$, $y$ and $z$, in the underwritten Equations.*

$\frac{xyz}{x+y}=200$, $\frac{xyz}{x+z}=150$, $\frac{xyz}{y+z}=120$.

Firſt, by multiplication,

$xyz=200x+200y=150x+150z=120y+120z$:

therefore, by reduction,

$$\left.\begin{array}{r} 50x+200y=150z \\ 200x+80y=120z \end{array}\right\} \text{ or } \left\{\begin{array}{l} x+4y=3z, \\ 5x+2y=3z. \end{array}\right.$$

Hence $x+4y=5x+2y$, and therefore $y=2x$:

which, ſubſtituted in $3z=x+4y$, gives $3z=9x$; and $z=3x$.

And, by ſubſtituting for both $y$ and $z$, in the equation $\frac{xyz}{x+y}=200$, we get $\frac{x\times 2x\times 3x}{x+2x}=200$;

that is, $2x^2=200$.

Conſequently $x=10$, $y=20$, and $z=30$.

## QUESTION LXXVI.

*To find the Ratio of two Numbers, whoſe Rectangle is equal to the Square of their Difference.*

Let the leſſer number be to the greater as 1 is to $x$; then, if the ſaid leſſer number be denoted by $z$, the greater will be expreſſed by $xz$, and we ſhall have $xz\times z=\overline{xz-z}^2$, or $xz^2=x^2z^2-2xz^2+z^2$ *(by the queſtion)*. Whence, dividing the whole by $z^2$, there reſults $x=x^2-2x+1$.

Therefore $x^2-3x=-1$;

and conſequently $x=\frac{3+\sqrt{5}}{2}=2.618$ &c.

Hence

Digitized by Google

Hence the ratio of any two numbers, whose rectangle is equal to the square of their difference, must be that of 2.618 &c. to unity.

## QUESTION LXXVII.

*To find two Numbers, whose Product is* 300 (a); *so that, if* 10 (b) *be added to the Lesser, and* 8 (c) *subtracted from the Greater, the Product of the Sum and Remainder shall, also, be equal to* 300 (a).

Let the greater number be denoted by $x$, and the lesser by $y$;

then will $\left\{\begin{matrix} xy = a \\ x-c \times y+b = a \end{matrix}\right\}$ *by the problem.*

By the last of which $xy+bx-cy-cb=a$.

From whence, the first equation bing subtracted, there rests $bx-cy-cb=0$:

therefore $bx=cb+cy$, and $x=\frac{cb+cy}{b}$: which substituted in the first equation, gives $\frac{cby+cy^2}{b}=a$.

From which we have $y^2+by=\frac{ab}{c}$:

and, by compleating the square, $y^2+by+\frac{bb}{4}=\frac{ab}{c}+\frac{bb}{4}$.

Whence $y=\sqrt{\frac{ab}{c}+\frac{bb}{4}}-\frac{b}{2}=15$;

and $x\left(=\frac{a}{y}\right)=20$.

## QUESTION LXXVIII.

*To divide* 100 *into two such Parts, that their Difference may be to their Sum, as their Rectangle to the Difference of their Squares.*

Let $a$ represent half the given number, and $x$ half the difference of its two parts; then, the greater of



Digitized by Google

them being expreſſed by $a+x$, and the leſſer by $a-x$, it will be $2x : 2a :: \overline{a+x \times a-x} : \overline{a+x}^2 - \overline{a-x}^2$; that is, by reduction, $x : a :: aa-xx : 4ax$:
hence $4ax^2 = a \times aa-xx$,
or $4x^2 = a^3 - x^2$,
and therefore $x = \sqrt{\frac{aa}{5}} = 22,36$.
So that, the greater part is 72,36; and the leſſer 27,64.

## QUESTION LXXIX.

*To divide the Number* 60 (a) *into two ſuch Parts, that their Product may be to the Sum of their Squares, in the Ratio of* 2 (m) *to* 5 (n).

Let $x$ be the greater part:
then, the leſſer part will be $a-x$,
the product $ax-xx$,
and the ſum of the ſquares $a^2-2ax+2x^2$.
Therefore $m : n :: ax-xx : a^2-2ax+2x^2$;
and ſo, $2mx^2-2max+ma^2 = nax-nx^2$.
Whence, $2mx^2+nx^2-2max-nax = -ma^2$;
that is, $\overline{2m+n} \times x^2 - \overline{2m+n} \times ax = -ma^2$;
or, $x^2-ax = -\frac{maa}{2m+n}$.
Therefore $x^2 - ax + \frac{aa}{4} = \left(\frac{aa}{4} - \frac{ma^2}{2m+n}\right) = \frac{aa}{4} \times \frac{n-2m}{n+2m}$,
and $x = \frac{a}{2}\sqrt{\frac{n-2m}{n+2m}} + \frac{a}{2} = 40$.

## QUESTION LXXX.

*To find two Numbers whoſe Product ſhall be* 320 (a), *and the Difference of their Cubes to the Cube of their Difference, as* 61 (n) *is to Unity.*

Let $x$ be the greater number, and $y$ the leſſer;
then will $xy = a$,
and $x^3 - y^3 : \overline{x-y}^3 :: n : 1$, *by the queſtion.*

Which

Which, by actually involving $x-y$,
becomes $x^3-y^3 : x^3-3x^2y+3xy^2-y^3 :: n : 1$.
From whence (by ſubtracting the conſequents from their antecedents)
we get $3x^2y-3xy^2 : x^3-3x^2y+3xy^2-y^3 :: n-1 : 1$;
or, which is the ſame, $3xy \times x-y : \overline{x-y}^3 :: n-1 : 1$.
Whence, dividing by $x-y$,
we have $3xy : \overline{x-y}^2 :: n-1 : 1$;
or, $3a : \overline{x-y}^2 :: n-1 : 1$ (becauſe $xy=a$).
From whence $\overline{x-y}^2 = \frac{3a}{n-1}$;

and conſequently $x-y = \sqrt{\frac{3a}{n-1}}$; which put $=b$.
Then from the equations $xy=a$, and $x-y=b$, the value of $x$ will be found $= \frac{\sqrt{b^2+4a}+b}{2} = 20$; and that of $y = \frac{\sqrt{b^2+4a}-b}{2} = 16$, *vid. Prob.* 65.

*The ſame otherwiſe.*

Let $z$ denote the half ſum, and $x$ the half difference, of the two numbers; then the greater will be expreſſed by $z+x$, and the leſſer by $z-x$; and we ſhall therefore have $\left\{\begin{array}{l} \overline{z+x} \times \overline{z-x} = a \\ \overline{z+x}^3 - \overline{z-x}^3 = n \times \overline{2x}^3 \end{array}\right\}$ *by the queſtion.*

that is $\left\{\begin{array}{l} z^2-x^2=a, \\ 6z^2x+2x^3=8nx^3. \end{array}\right.$

The laſt of which, divided by $2x$,
gives $3z^2+x^2=4nx^2$.
From whence, the treble of the firſt being deducted,
we have $4x^2=4nx^2-3a$; and conſequently $x = \sqrt{\frac{3a}{4n-4}} = 2$,
Hence $z \ (= \sqrt{a+xx}) = 18$; therefore $z+x=20$,
and

and $z-x=16$; which are the two numbers that were to be found.

## QUESTION LXXXI.

*A Farmer received 7l. 4s. for a certain Quantity of Wheat, and an equal Sum, at a Price leſs by 1s. 6d. a Buſhel, for a Quantity of Barley, which exceeded that of the Wheat by 16 Buſhels. How many Buſhels were there of each?*

Put $a$=the total value of each ſort of grain,
$b$=the difference of the quantities,
$c$=the difference of the prices *per* buſhel,
and $x$=the number of buſhels of the wheat.

Then, dividing the whole price by the number of buſhels, we have $\frac{a}{x}$ for the price of the wheat *per* buſhel: and, in the ſame manner, the price of the barley *per* buſhel will appear to be $\frac{a}{x+b}$.

Therefore $\frac{a}{x}-\frac{a}{x+b}=c$, *by the queſtion.*

Whence (by reduction) $ax+ab-ax=cx^2+bcx$;

or, $\frac{ab}{c}=x^2+bx$.

Conſequently $x=\sqrt{\frac{ab}{c}+\frac{bb}{4}}-\frac{b}{2}=32$, the number of buſhels of wheat; and $x+16=48$, the number of buſhels of barley.

QUESTION

Digitized by Google

## QUESTION LXXXII.

*One bought two Pieces of Cloth of different Sorts, whereof the finer coſt 4 Shillings a Yard more than the other; ſo that, for the fineſt, he paid 360 Shillings; whereas the coarſeſt, which exceeded the fineſt by 10 Yards, coſt him, only, 320 Shillings. How many Yards were there of each Piece?*

Let $x$ be the number of yards of the fineſt, and $y$, the number of ſhillings a yard: then $x+10$, will be the length of the coarſeſt piece, and $y-4$, its price *per* yard.

Hence $\left\{\begin{array}{l} xy = 360 \\ x+10 \times y-4=320 \end{array}\right\}$ *by the queſtion.*

By the laſt equation $xy+10y-4x=360$; from whence deducting $xy=360$, we have $10y-4x=0$.

Therefore $10y=4x$; and $y=\frac{2x}{5}$. Which, being ſubſtituted in the firſt equation, we get $\frac{2x^2}{5}=360$.

Whence $x$ comes out $=\sqrt{900}=30$.

## QUESTION LXXXIII.

*There are two Numbers, whoſe Rectangle is equal to the Difference of their Squares; and the Sum of their Squares is alſo equal to the Difference of their Cubes. What are thoſe Numbers?*

Let $x$ denote the leſſer number, and let the greater be in proportion thereto as $y$ is to unity; or, which is the ſame thing, let the greater number be denoted by $xy$.

Then $\left\{\begin{array}{l} x \times xy=x^2y^2-x^2 \\ x^2y^2+x^2=x^3y^3-x^3 \end{array}\right\}$ *by the problem.*

Therefore $\left\{\begin{array}{l} y=y^2-1 \\ y^2+1=xy^3-x \end{array}\right\}$ *by diviſion.*

From

From the firſt of which equations, $y^2 - y = 1$; whence $y^2 - y + \frac{1}{4} = \frac{5}{4}$, and conſequently $y = \frac{1+\sqrt{5}}{2}$. But, by the ſecond equation, $x = \frac{yy+1}{yyy-1} = \frac{y+2}{2y}$ (becauſe $yy = y+1$) $= \frac{1}{2} + \frac{1}{y} = \frac{1}{2} + \frac{2}{\sqrt{5}+1} = \frac{1}{2} + \frac{2\sqrt{5}-2}{4}$ (by multiplying both the numerator and denominator by $\sqrt{5}-1$) $= \frac{\sqrt{5}}{2}$. Therefore the two quantities ſought are $\frac{1}{2}\sqrt{5}$, and $\frac{5+\sqrt{5}}{4}$.

## QUESTION LXXXIV.

*A ſets out from London for York, at the ſame Time as B ſets out from York for London; and the Rate at which they travel is ſuch, that* A, 9 *Hours after their meeting arrives at York, and* B *at London, in* 16 *Hours after. The Queſtion is, to find in what Time each Traveller performs his Journey.*

Let $x$ denote the number of hours travelled by each before the time of their meeting on the road.
Then, ſince A *(by the former part of the queſtion)* goes over the very ſame ground in 9 *(a)* hours, as B travelled in $x$ hours, we have, therefore, as $a : x :: x : \frac{xx}{a}$, the time wherein B travels a diſtance equal to that gone over by A, in $x$ hours: but *(by the latter part of the queſtion)* B, in 16 *(b)* hours, travels the very ſame diſtance as A in $x$ hours. Hence it is evident that $\frac{xx}{a}$ and $b$ are equal to each other:

and conſequently that $x = \sqrt{ab} = 12$.

Therefore

Digitized by Google

Therefore A performs the journey in 21 hours, and B in 28 hours.

## QUESTION LXXXV.

*The Sum of* 190*l. was divided between three Perſons; whoſe Shares were in Geometrical Proportion, and the greateſt of them exceeded the leaſt by* 50*l. What were all the ſeveral Shares?*

If $x$ be put for the leaſt of them, then the greateſt will be $x+50$; the ſum of which two, ſubtracted from (190) the whole, leaves $140-2x$ for the mean ſhare. Therefore, $x : 140-2x :: 140-2x : x+50$, *by the queſtion.*

And conſequently $\overline{140-2x}^2 = x \times \overline{x+50}$;

that is, $19600-560x+4x^2 = x^2+50x$.

Whence $3x^2-610x = -19600$,

and $x^2-\frac{610x}{3} = -\frac{19600}{3}$.

Hence, by compleating the ſquare,

$x^2-\frac{610x}{3}+\frac{93025}{9}\left(=-\frac{19600}{3}+\frac{93025}{9}\right)=\frac{34225}{9}$;

and, by extracting the root, $x-\frac{305}{3}=\pm\frac{185}{3}$.

Therefore $x=\frac{305-185}{3}=40$: and ſo the other two ſhares are 60 and 90 pounds.

## QUESTION LXXXVI.

*Two Notes, one of* 120*l. payable in* 6 *Months, and the other of* 150*l. payable in* 9 *Months, were diſcounted for* 8*l.* 10*s. What Rate of Intereſt were they diſcounted at?*

Let $x$ denote the intereſt of one pound for 12 months. Then the amount of 1*l.* in 6 months being $1+\frac{x}{2}$, and,



Digitized by Google

in 9 months $1+\frac{3x}{4}$ the *present value* of the bill, due at the end of 6 months, will therefore be $\frac{120}{1+\frac{1}{2}x}$; and *that* of the bill, due at the end of 9 months, $\frac{150}{1+\frac{3}{4}x}$.

Whence, we have $\frac{120}{1+\frac{1}{2}x}+\frac{150}{1+\frac{3}{4}x}$ $(=120+150-8.5)$ $=261,5$ *(by the question.)*

Which, by reduction, becomes $120+90x+150+75x=261,5\times 1+\frac{1}{4}x+\frac{1}{8}x^2$;

or, $270+165x=\frac{261,5\times 8+10x+3x^2}{8}$.

Therefore $\frac{270+165x\times 8}{261,5}=3x^2+10x+8$,

that is, $\frac{2160}{261,5}+\frac{1320x}{261,5}=3x^2+10x+8$,

From which we have

$x^2+\frac{2590}{3\times 523}\times x=\frac{136}{3\times 523}$,

Whence, by compleating the square, &c.

$x$ is found $=\sqrt{\frac{136}{1569}+\left|\frac{1295}{1569}\right|^2}-\frac{1295}{1569}=$

$\frac{\sqrt{136\times 1569+1295\times 1295}-1295}{1569}=0,05093$. Which multiplied by 100, gives 5,093*l.* or 5*l.* 1*s.* 10½*d.* nearly, for the rate *per cent.* at which the notes were discounted.

QUESTION

## QUESTION LXXXVII.

A *and* B *take, in Trade,* 5940*l.* per Annum; *each; but* A, *whoſe Profits are* 2 per Cent. *greater than thoſe of* B, *clears* 100*l.* per Annum *more than* B. *What are the Profits of each,* per Cent? *And what do they clear* per Annum?

Let $c=100$ (pounds),
$b$ $(=5940)$ =the whole ſum taken by each,
$d$ $(=100l.)$ =the given difference of their gains,
$x$=what A gains *per cent.*
$x-a$ $(=x-2)$ =what B gains *per cent.*

Now, ſince A, in taking $c+x$ pounds, gains $x$ pounds, it will be $c+x : x :: b : \frac{bx}{c+x}$, the whole gain of A:

and, in the ſame manner, we have,

$c+x-a : x-a :: b : \frac{b \times x-a}{c+x-a}$, the whole gain of B:

therefore $\frac{bx}{c+x} - \frac{b \times x-a}{c+x-a} = d$, *by the queſtion.*

Hence $abc = x^2 + 2cx - ax - ac + cc \times d$;

and conſequently $x^2 + \overline{2c-a} \times x = \frac{abc}{d} + ac - cc$.

Whence, by compleating the ſquare,

$x^2 + \overline{2c-a} \times x + \left(\frac{2c-a}{2}\right)^2 = \frac{abc}{d} + \frac{aa}{4}$;

and therefore $x = \sqrt{\frac{abc}{d} + \frac{aa}{4}} - \frac{2c-a}{2} = 10$.

From which it appears that A gained 10 *per cent.* and cleared 540*l. per annum*; and that B gained 8 *per cent.* and cleared 440*l. per annum.*



Digitized by Google

## QUESTION LXXXVIII.

*Of four Numbers in Geometrical Progreſſion, there is given the Sum of the two leaſt* =20 (a) *and the Sum of the two greateſt* =45 (b); *to find the Numbers.*

Let $x$ denote the firſt number, and $y$ the third; then the ſecond being expreſſed by $a-x$, and the fourth by $b-y$, the four terms of the progreſſion, placed in order, will ſtand thus, $x$, $a-x$, $y$, $b-y$.
Whence, by the nature of proportionals,

we have $\begin{cases} x \times \overline{b-y} = \overline{a-x} \times y, \\ xy = \overline{a-x}^2. \end{cases}$

From the firſt of which equations $y$ is $=\frac{bx}{a}$: and, by ſubſtituting in the ſecond, we have $\frac{bx^2}{a} = \overline{a-x}^2$.
Whence, by extracting the ſquare root on both ſides,

$x\sqrt{\frac{b}{a}} = a-x$;

and conſequently $x = \frac{a}{1+\sqrt{\frac{b}{a}}} = 8$.

Therefore 8, 12, 18, and 27, are the four numbers that were to be found.

## QUESTION LXXXIX.

*The Sum of* 700*l.* (a) *was divided among four Perſons,* A, B, C *and* D; *whoſe Shares were in Geometrical Progreſſion; and the Difference between the greateſt and leaſt, was to the Difference between the two Means, as* 37 (m) *to* 12 (n). *What were all the ſeveral Shares?*

Let the ſhare of A, or the firſt term of the progreſſion, be denoted by $x$, and let the common ratio be that of 1 to $y$.

Then

Then $x+xy+xy^2+xy^3=a$
and $xy^3-x : xy^2-xy :: m : n$ } *by the queſtion.*
From which proportion we have
$y^3-1=\overline{y-1}\times\frac{my}{n}$, or $y^2+y+1=\frac{my}{n}$ (by dividing the whole by $y-1$).
Hence $y$ is found $=\frac{m-n+\sqrt{mm-2mn-3nn}}{2n}=\frac{25+7}{24}$
$=\frac{4}{3}$. Whence $x\left(=\frac{a}{1+y+y^2+y^3}\right)$ is given $=$
$\frac{27\times 700}{27+36+48+64}=108$. Therefore the four ſhares are
108, 144, 192, and 256*l.*

## QUESTION XC.

*Of four Numbers in Arithmetical Progreſſion, the Sum of the Squares of the two Means is given* $=400$ (a); *and the Sum of the Squares of the two Extremes* $=464$ (b). *To determine the Numbers.*

If $x$ be put for the leſſer extreme, and $y$ for the common difference; then the four numbers will be expreſſed by,
$x$, $x+y$, $x+2y$, and $x+3y$ reſpectively; and we ſhall therefore have $\left\{\begin{matrix}\overline{x+y}^2+\overline{x+2y}^2=a\\ x^2+\overline{x+3y}^2=b\end{matrix}\right\}$ or

$\left\{\begin{matrix}2x^2+6xy+5y^2=a\\ 2x^2+6xy+9y^2=b\end{matrix}\right\}$ *by the queſtion.*
Whence, by ſubtraction, $4y^2=b-a$:
and therefore $y=\frac{\sqrt{b-a}}{2}=4$.
But, by the firſt equation, $x^2+3xy=\frac{a}{2}-\frac{5yy}{2}$:
from which (looking upon $y$ as known) we get
$x=\sqrt{\frac{a}{2}-\frac{yy}{4}}-\frac{3y}{2}=8$.

Therefore

Therefore 8, 12, 16, and 20, are the four numbers that were to be found.

## QUESTION XCI.

*The Sum of four Numbers, in Arithmetical Progreſſion, being given* $=56$ (b), *and the Sum of their Squares* $=$ 864 (c); *to find the Numbers.*

If half the ſum of the two middle numbers be denoted by $a$, and half their difference by $x$, the numbers themſelves will be expreſſed by $a-x$, and $a+x$: and we ſhall have

$$\left\{\begin{array}{l}\overline{a-3x}+\overline{a-x}+\overline{a+x}+\overline{a+3x}=b \\ \overline{a-3x}^2+\overline{a-x}^2+\overline{a+x}^2+\overline{a+3x}^2=c\end{array}\right\}\text{by the nature of the queſtion.}$$

Hence, by reduction, $4a=b$, and $4aa+20xx=c$.

Therefore $a=\frac{b}{4}=14$; and $x=\sqrt{\frac{c}{20}-\frac{aa}{5}}=2$.

From which the numbers themſelves are given; being 8, 12, 16 and 20.

By the ſame way of proceeding the problem may be reſolved, when the progreſſion is ſuppoſed to conſiſt of 6, 8, 10, or any other, even, number *(n)* of terms.

For the ſum of the ſquares of the two means ($a-x$, and $a+x$) being $=2\times\overline{aa+xx}$; and the ſum of the ſquares of the two terms ($a-3x$ and $a+3x$) next adjacent to them $=2\times\overline{aa+9xx}$; alſo the ſum of the ſquares of the two terms ($a-5x$ and $a+5x$) next adjacent to theſe laſt being $=2\times\overline{aa+25xx}$, &c. &c. it follows that

$2\times\overline{aa+xx}+2\times\overline{aa+9xx}+2\times\overline{aa+25xx}+$ &c. $=c$.

Which equation, by putting $1+9+25+36+$ &c. (continued to $\frac{n}{2}$ terms) $=f$, is reduced to $naa+2fxx=c$.

Whence

Whence $x=\sqrt{\frac{c-naa}{2f}}$; from which, as $a$ is given $=\frac{b}{n}$, the value of $x$ will alſo be known.

## QUESTION XCII.

*Having the Sum* (b) *and the Sum of the Squares* (c) *of five Numbers, in Arithmetical Progreſſion; to determine the Progreſſion.*

Let $a$ denote the middle number, and $x$ the common difference; then the five terms of the progreſſion will be, $a-2x$, $a-x$, $a$, $a+x$, and $a+2x$; whence we have $5a=b$, and $5aa+10xx=c$, by the conditions of the queſtion. From which $a$ is found $=\frac{b}{5}$, and $x$ $(=\sqrt{\frac{c}{10}-\frac{aa}{2}})=\sqrt{\frac{c}{10}-\frac{bb}{50}}$.

After the ſame manner the problem may be inveſtigated, when any odd number ($n$) of terms is propounded: for the ſum of the ſquares of the 2 terms ($a-x$ and $a+x$) adjacent to the middle one being $=2\times\overline{aa+xx}$; and that of the two terms ($a-2x$ and $a+2x$) next to theſe $=2\times\overline{aa+4xx}$, &c. &c. it is evident therefore that

$aa+2\times\overline{aa+xx}+2\times\overline{aa+4xx}+2\times\overline{aa+9xx}+$ &c. $=c$.

Which equation, by putting $1+4+9+16+$ &c. (to $\frac{n-1}{2}$ terms) $=f$, becomes $naa+2fxx=c$.

Hence $x=\sqrt{\frac{c-naa}{2f}}$; from which, as $a$ is given $=\frac{b}{n}$, all the terms of the progreſſion will be known.

By a like proceſs, if, inſtead of the ſum of the ſquares, the ſum of the cubes, or biquadrates, be given, the problem may be reſolved.

QUESTION

Digitized by Google

## QUESTION XCIII.

*A Traveller, bound to a certain Place at the Diſtance of* 140 *Miles, goes* 26 *Miles the firſt Day,* 24 *Miles the ſecond Day,* 22 *Miles the third; and ſo on, in Arithmetical Progreſſion, decreaſing* 2 *Miles every Day. In how many Days will he arrive at the End of his Journey?*

Put $b=140$, the whole, given, diſtance,
$c=26$, the part thereof travelled in the firſt day,
$d=2$, the common difference by which each day's journey decreaſes,
and $x=$the number of days wherein the *whole* journey is performed.

Then, from the nature of the queſtion, it is evident, that, the laſt day's journey will fall ſhort of the firſt, by $x-1$ times the common difference $(d)$; and is, therefore, truly expreſſed by $c-\overline{x-1}\times d$.

But it is well known that the ſum of any arithmetical progreſſion is equal to the ſum of the firſt and laſt terms multiplied by half the number of the terms:

hence we have $\overline{c+c-\overline{x-1}\times d}\times\frac{x}{2}$ for the whole diſtance travelled; and conſequently this equation, $\overline{2c-\overline{x-1}\times d}\times\frac{x}{2}=b$.

Whence $2cx-dx^2+dx=2b$,

and $xx-\frac{2c+d}{d}\times x=-\frac{2b}{d}$.

From which $x=\pm\sqrt{\frac{\overline{2c+d}^2}{4dd}-\frac{2b}{d}}+\frac{2c+d}{2d}=7$.

QUESTION

## QUESTION XCIV.

*After* A, *who travelled at the Rate of* 4 *Miles an Hour, had been ſet out two Hours and three Quarters,* B *ſet out to overtake him; and in order thereto went* $4\frac{1}{2}$ *Miles the firſt Hour,* $4\frac{3}{4}$ *the ſecond,* 5 *the third; and ſo on, gaining a Quarter of a Mile every Hour. How many Hours did it take him to come up with* A?

Put $a=4$, the diſtance travelled by A, *per hour.*
$b=11$, the diſtance gone by A before B ſet out,
$c=4\frac{1}{2}$, the miles travelled by B in the firſt hour,
$d=\frac{1}{4}$ of a mile, the common exceſs,
and $x=$the number of hours required.

Then it is evident (from the preceding problem) that the diſtance travelled by B, in the laſt hour, will be $c+\overline{x-1}\times d$; and that $2c+\overline{x-1}\times d\times\frac{x}{2}$ will expreſs the whole diſtance travelled by B in $x$ hours.

But the diſtance of A in $x$ hours being $ax$, the whole diſtance travelled by A will therefore be $ax+b$; which being equal to *that* of B *(by the queſtion)* we thence have,

$2c+\overline{x-1}\times d\times\frac{x}{2}=ax+b$;

and conſequently $2cx+dx^2-dx=2ax+2b$;

or $xx+\frac{2c-2a-d}{d}\times x=\frac{2b}{d}$.

Which (by making $\frac{2c-2a-d}{d}=f$) gives

$x=\sqrt{\frac{2b}{d}+\frac{ff}{4}}-\frac{f}{2}=8$, the number of hours required.



Digitized by Google

## QUESTION XCV.

*To find four Numbers in Arithmetical Progreſſion, which being increaſed by* 2, 4, 8 *and* 15, *reſpectively, the Sums ſhall be in Geometrical Progreſſion.*

Let $x$ denote the leaſt number, and $y$ the common difference. Then the four numbers will be expreſſed by $x$, $x+y$, $x+2y$, and $x+3y$;
and the four ſpecified ſums,
by $x+2$, $x+y+4$, $x+2y+8$, and $x+3y+15$.
Whence, by the nature of geometrical proportionals,
we have $\begin{cases} x+2 \times x+2y+8 = x+y+4^2, \\ x+2 \times x+3y+15 = x+y+4 \times x+2y+8, \end{cases}$
that is $\left\{\begin{array}{l} y^2+4y=2x \\ 2y^2+10y+2=5x \end{array}\right\}$ *by multiplication, and tranſpoſition.*
Hence $5y^2+20y=4y^2+20y+4$;
therefore $y^2=4$, and $y=2$:
From which $x=6$; ſo that the other three numbers are 8, 10, and 12 reſpectively.
For 6, 8, 10 and 12 are in arithmetical progreſſion; and $6+2$, $8+4$, $10+8$, and $12+15$, are in geometrical progreſſion.

## QUESTION XCVI.

*The Rectangle* (a) *and the Sum of the Cubes* (*b*) *of two Numbers being given; to determine the Numbers.*

Let $x$ denote the greater, and $y$ the leſſer number:
then will $\left\{\begin{array}{l} xy=a \\ x^3+y^3=b \end{array}\right\}$ *by the queſtion.*
Whence, by involution, $x^3y^3=a^3$,
and $x^6+2x^3y^3+y^6=b^2$.
Let the quadruple of the former of theſe two equations be ſubtracted from the latter, and you will have $x^6-2x^3y^3+y^6=b^2-4a^3$;

and,

Digitized by Google

and, by extracting the square root, on both sides, $x^3-y^3=\sqrt{b^2-4a^3}$. Which, added to the first equation gives $2x^3=b+\sqrt{b^2-4a^3}$.

Consequently $x=\sqrt[3]{\frac{1}{2}b+\frac{1}{2}\sqrt{bb-4a^3}}$;

and $y\left(=\frac{a}{x}\right)=\sqrt[3]{\frac{a}{\frac{1}{2}b+\frac{1}{2}\sqrt{bb-4a^3}}}$.

*The same otherwise.*

Since $xy=a$, we have $y=\frac{a}{x}$; and therefore

$x^3+\frac{a^3}{x^3}=b$, by the question.

Whence $x^6+a^3=bx^3$,

or, $x^6-bx^3=-a^3$.

Therefore, by compleating the square,

$x^6-bx^3+\frac{bb}{4}\left(=\frac{bb}{4}-a^3\right)=\frac{bb-4a^3}{4}$;

and, by extracting the root, $x^3-\frac{b}{2}=\frac{\sqrt{bb-4a^3}}{2}$.

Hence $x^3=\frac{1}{2}b+\frac{1}{2}\sqrt{bb-4a^3}$;

and consequently the values of $x$ and $y$, *the very same as before.*

From either of the above solutions, a general theorem, for the resolution of cubic equations (according to the manner of *Cardan)* is very easily deducible.

For, by putting $z=x+y$, and involving each side to the third power, we have $z^3=x^3+3x^2y+3xy^2+y^3=x^3+y^3+3xy\times\overline{x+y}=x^3+y^3+3a\times z$ (by substituting for $xy$ and $x+y$, their equals, $a$ and $z$). From whence, by transposition,

$z^3-3az\ (=x^3+y^3)\ =b$.

But it appears from above, that $z\ (=x+y)$



Digitized by Google

is $=\sqrt[3]{\frac{1}{2}b+\frac{1}{2}\sqrt{bb-4a^3}}+\sqrt[3]{\frac{1}{2}b+\frac{1}{2}\sqrt{bb-4a^3}}$ : $a$

Which value, therefore, is the true root of the equation $z^3-3az=b$.

From which the root of the equation $z^3+cz=b$ (where the second term is positive) will also be given, by assuming $-3a=c$, and substituting for $a$: whence $z$ is found

$$=\sqrt[3]{\frac{b}{2}+\sqrt{\frac{bb}{4}+\frac{c}{3}\Big|^3}}-\sqrt[3]{\frac{b}{2}+\sqrt{\frac{bb}{4}+\frac{c}{3}\Big|^3}}. \quad \tfrac{1}{3}c$$

## QUESTION XCVII.

*The Difference of two Numbers being given* $=4$, *and the Sum of their Cubes* $=2240$; *to determine the Numbers.*

Let $x$ denote their half sum, and $d$ their half difference, then the greater being $x+d$, and the lesser $x-d$, we have

$\overline{x+d}|^3+\overline{x-d}|^3=2240$,

that is $2x^3+6d^2x=2240$,

or, $x^3+3d^2x=\frac{2240}{2}$.

Put $c=3d^2$ $(=12)$, and $b=\frac{2240}{2}$ $(=1120)$

then it will become $x^3+cx=b$:

from whence by the general theorem, in the last problem, $x=$

$$\sqrt[3]{\frac{b}{2}+\sqrt{\frac{bb}{4}+\frac{c}{3}\Big|^3}}-\sqrt[3]{\frac{b}{2}+\sqrt{\frac{bb}{4}+\frac{c}{3}\Big|^3}}=10. \quad \tfrac{1}{3}c$$

Thefore 12 $(=x+d)$ and 8 $(=x-d)$ are the two numbers that were to be found.

QUESTION

Digitized by Google

## QUESTION XCVIII.

*It is propoſed to divide the Number* 24 (2*a*) *into two ſuch Parts, that the Difference of their Cubes may be* 3584 (2*b*).

Let $a+x$ expreſs the greater part, and $a-x$ the leſſer;
then will $\overline{a+x}|^3-\overline{a-x}|^3=2b$;
that is, $6a^2x+2x^3=2b$;
or $x^3+3a^2x=b$.

Put $c=3a^2$, and $\sqrt[3]{\frac{b}{2}+\sqrt{\frac{bb}{4}+\frac{c}{3}}^3}=r$.

Then will $x^3+cx=b$,

and $x=r-\frac{\frac{1}{3}c}{r}=4$, *by the theorem in the preceding page.*

Whence 8 and 16 are the two required numbers.

## QUESTION XCIX.

*The Sum of the Squares of two Numbers being given* =208 (g) *and the Sum of their Cubes* =2240 (h); *to find the Numbers.*

Let the greater number be denoted by $x+y$, and the leſſer by $x-y$:

then will $\left\{\begin{matrix}2x^2+2y^2=g\\2x^3+6xy^2=b\end{matrix}\right\}$ *by the queſtion.*

From the firſt of theſe equations, multiplied by $3x$, let the ſecond be ſubtracted;
whence you will have $4x^3=3gx-b$;
and conſequently $x^3-\frac{3gx}{4}+\frac{b}{4}=0$;
that is, in numbers, $x^3-156x+560=0$;
the roots of which equation (by Sect. 12 of my Treatiſe of Algebra) will be found to be 10,4, and $-14$; whereof the firſt, only, is for our purpoſe: by means of which we get $y=(\sqrt{\frac{1}{2}g-x^2})=2$.

Therefore

Digitized by Google

Therefore 12 and 8 are the two numbers that fulfil the conditions of the problem.

Note. The reſolution of the above equation, by the theorem referred to in the preceding examples, is impoſſible; becauſe the ſquare root of a negative quantity is to be extracted; as, upon trial, will be found.

## QUESTION C.

*The Sum* (a) *and the continual Product* (b) *of four Numbers, in Geometrical Progreſſion, being given; to determine the Numbers.*

If the leſſer of the two means be repreſented by $x-y$, and the greater by $x+y$, we ſhall have

$$\left.\begin{array}{l} \frac{\overline{x-y}^2}{x+y}+\overline{x-y}+\overline{x+y}+\frac{\overline{x+y}^2}{x-y}=a \\ \text{and } \overline{x-y}^2\times\overline{x+y}^2=b \end{array}\right\} \textit{by the queſtion.}$$

From the ſecond of theſe equations, by putting $c=\sqrt{b}$, and extracting the ſquare root, there comes out

$$\overline{x-y}\times\overline{x+y}\ (=xx-yy)=c,$$

Moreover, from the firſt, we get

$$\overline{x-y}^3+2x\times\overline{x-y}\times\overline{x+y}+\overline{x+y}^3=a\times\overline{x-y}\times\overline{x+y};$$

or, $2cx+\overline{x-y}^3+\overline{x+y}^3=ac$ (by ſubſtitution)

that is, $2cx+2x^3+6xy^2=ac$;

or $2cx+2x^3+6x\times\overline{xx-c}=ac$ (becauſe $yy=xx-c$,)

Hence $x^3-\frac{cx}{2}=\frac{ac}{8}$. From which, by the theorem in page 60, the value of $x$ may be found.

QUESTION

Digitized by Google

## QUESTION CI.

*The Compound Intereſt of a certain Sum of Money, put out for 4 Years, amounted to 344l. $\frac{1}{800}$; but the ſimple Intereſt thereof, for the ſame Time, and at the ſame Rate, would have been only 320l. What was the Sum put out? and what the Rate of Intereſt?*

Put $a=344.81$, and $b=320$; and let $x$ denote the intereſt of 1*l.* for one year.

Therefore, ſince the ſimple intereſt of 1*l.* for 4 years is $4x$, and the compound intereſt $\overline{1+x}^4-1$, or $4x+6x^2+4x^3+x^4$,

we have, as $4x+6x^2+4x^3+x^4 : 4x :: a : b$, by the nature of the queſtion:

and conſequently $6x+4x^2+x^3=\frac{4a-4b}{b}=0.310125$.

From the reſolution of which equation, $x$ will be found $=.05$. Therefore the rate of intereſt was 5 *per cent.* and the ſum put out 1600*l.*

## QUESTION CII.

*The Sum* (s) *and the Product* (p) *of any two Numbers being given, to find the Sum of the Squares, Cubes, Biquadrates, &c. of thoſe Numbers.*

Let the two numbers be denoted by $x$ and $y$:

then $x+y=s$ and $xy=p$ } by the queſtion.

The former of which equations, ſquared,

gives $x^2+2xy+y^2=s^2$; from whence the double of the latter being ſubtracted,

we get $x^2+y^2=s^2-2p$, *the ſum of the ſquares.*

Let this equation be multiplied by $x+y=s$,

and there ariſes $x^3+xy^2+yx^2+y^3=s^3-2sp$,

or, $x^3+xy\times x+y+y^3=s^3-2sp$,

that is, $x^3+p\times s+y^3=s^3-2sp$, (becauſe $xy=p$, and $x+y=s$).

Therefore

Therefore $x^3+y^3=s^3-3sp$, *the sum of the cubes.*
Again, multiply this laſt equation by $x+y=s$,
and you will have $x^4+xy\times x^2+y^2+y^4=s^4-3s^2p$,
or $x^4+p\times s^2-2p+y^4=s^4-3s^2p$ (becauſe $x^2+y^2=s^2-2p$.)
Whence $x^4+y^4=s^4-4s^2p+2p^2$, *the ſum of the fourth powers.*
Multiply, again, by $x+y=s$, and you will have
$x^5+xy\times x^3+y^3+y^5=s^5-4s^3p+2sp^2$,
or, $x^5+p\times s^3-3sp+y^5=s^5-4s^3p+2sp^2$,
and therefore $x^5+y^5=s^5-5s^3p+5sp^2$, *the ſum of the fifth powers.*
From whence the law of continuation is manifeſt; being ſuch, that the ſum of the next ſuperior powers will, *always*, be obtained by multiplying the ſum of the powers laſt found by $s$, and ſubtracting the ſum of the preceding ones drawn into $p$, from the product.
So that the ſum of the $n^{th}$ powers (expreſſed in a general manner)
or $x^n+y^n$, will be $=s^n-ns^{n-2}p+n\times\frac{n-3}{2}s^{n-4}p^2-n\times\frac{n-4}{2}\times\frac{n-5}{3}s^{n-6}p^3+n\times\frac{n-5}{2}\times\frac{n-6}{3}\times\frac{n-7}{4}s^{n-8}p^4-$ &c.

## QUESTION CIII.

*The Sum* (a) *and the Sum of the Squares* (b) *of four Numbers, in Geometrical Progreſſion being given; to find the Numbers.*

If $x$ and $y$ be aſſumed to repreſent the two middle numbers, then, from the nature of continued proportionals, the two extremes will be expreſſed by $\frac{xx}{y}$ and $\frac{yy}{x}$: and ſo
we ſhall have $\frac{x^2}{y}+x+y+\frac{y^2}{x}=a$,
and $\frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=b$.

Put

Put $x+y=u$, and $xy=z$:
then, from the firſt equation, $\frac{x^2}{y}+\frac{y^2}{x}=a-u$,
and therefore $x^3+y^3=xy\times a-u$.
But, becauſe the ſum of the two means ($x$ and $y$) is expreſſed by $u$, and their rectangle by $z$, it is evident, *from queſtion* 102, that the ſum of their ſquares ($x^2+y^2$) will be exhibited by $u^2-2z$, and the ſum of their cubes ($x^3+y^3$) by $u^3-3uz$.
And, in like manner, the ſum of the two extremes $\left(\frac{x^2}{y}+\frac{y^2}{x}\right)$ being denoted by $a-u$, and their rectangle by $z$, the ſum of their ſquares will be $=\overline{a-u}^2-2z$, and the ſum of their cubes $=\overline{a-u}^3-\overline{3a-3u}\times z$ (but this laſt is of no uſe in the preſent caſe.)
Hence, by ſubſtituting theſe ſeveral values in our ſecond, and laſt equations,
we get $\overline{a-u}^2-2z+u^2-2z=b$,
and $u^3-3uz=z\times a-u$;
that is, by reduction,
$2u^2-2au+a^2-b=4z$,
and $u^3=2u+a\times z$;
and, conſequently, $\overline{2u+a}\times\overline{2u^2-2au+a^2-b}=4u^3$:
whence, by reduction, $u^2+\frac{bu}{a}=\frac{aa-b}{2}$;
and therefore $u=\sqrt{\frac{aa-b}{2}+\frac{bb}{4aa}}-\frac{b}{2a}$.
From which the ſeveral values of $z$, $x$, $y$ will alſo become known.

## QUESTION CIV.

*The Sum* (a) *and the Sum of the Cubes* (c) *of four Numbers in continued Geometrical Proportion being given; to determine the Numbers.*

Let the notation of the *preceding problem* be retained; then our two equations *(in this caſe)*



Digitized by Google

will be $\frac{x^2}{y}+x+y+\frac{y^2}{x}=a$,

and $\frac{x^6}{y^3}+x^3+y^3+\frac{y^6}{x^3}=c$.

But, it appears, *from thence*, that the ſum of the cubes $(x^3+y^3)$ of the two means is $=u^3-3uz$; and that the ſum of the cubes $\left(\frac{x^2}{y}+\frac{y^2}{x}\right)$ of the two extremes is $=\overline{a-u}^3-3a-3u\times z$. Therefore our laſt equation, by ſubſtituting theſe values, becomes $u^3+\overline{a-u}^3-3az=c$. And, it appears, *by the laſt problem*, that the firſt equation (by a like ſubſtitution) is reduced to $u^3=\overline{2u+a}\times z$.

Hence, by exterminating $z$ out of the two equations thus derived, and putting, $\frac{a^2}{3}-\frac{c}{3a}=d$, we obtain

$\overline{2u+a}\times\overline{u^2-au+d}=u^3$;

or $u^3-au^2-\overline{aa-2d}\times u+ad=0$.

From whence, the value of $u$ being found, the reſt of the quantities will be known.

In the ſame manner the problem may be reſolved, when (inſtead of the ſum of the cubes) the ſum of the 4th, 5th, or 6th, *&c.* powers is given.

For the ſum of the $n^{th}$ powers of the two means (or $x^n+y^n$) being, *univerſally*, $=u^n-nu^{n-2}z+n\times\frac{n-3}{2}u^{n-2}z^2$ *&c.* (*See Queſtion* 102.)

and the ſum of the $n^{th}$ powers of the two extremes $=\overline{a-u}^n-n\times\overline{a+u}^{n-2}z+n\times\frac{n-3}{2}u^{n-4}z^2$, *&c.* (ſince the ſum of the roots is here $=a-u$);

we therefore have $u^n+\overline{a-u}^n-nz\times\overline{u^{n-2}+\overline{a-u}^{n-2}}+\frac{n}{1}\times\frac{n-3}{2}z^2\times\overline{u^{n-4}+\overline{a-u}^{n-4}}-$ *&c.* $=c$. Which equation, by writing $\frac{u^3}{2u+a}$ inſtead of its equal $z$, becomes

comes $u^n+\overline{a-u}^n-\frac{nu^3}{2u+a}\times\overline{u^{n-2}+\overline{a-u}^{n-2}}+\frac{n\times\overline{n-3}}{2}$ $\times\frac{u^6}{\overline{2u+a}^2}\times\overline{u^{n-4}+\overline{a-u}^{n-4}}+$ &c. $=c$. Whence $u$ may be found.

## QUESTION CV.

*Having given the Sum* (a) *and the Sum of the Squares* (b) *of five Numbers in Geometrical Progreſſion; to determine the Progreſſion.*

Let $x$, $z$, and $y$ denote the three middle numbers, taken in order: then $\frac{xx}{z}$ will be the firſt number, and $\frac{yy}{z}$ the laſt; and we ſhall have

$$\left.\begin{cases}\frac{xx}{z}+x+z+y+\frac{yy}{z}=a,\\ \frac{x^4}{z^2}+x^2+z^2+y^2+\frac{y^4}{z^2}=b,\end{cases}\right\}\textit{by the queſtion.}$$

Put $u=x+y$; then, from the firſt equation,

$\frac{xx}{z}+\frac{yy}{z}=a-u-z$.

Therefore, ſeeing the ſum of the two extremes is expreſſed by $a-u-z$, and their rectangle by $z^2$ (from the nature of proportionals) the ſum of their ſquares will be $=\overline{a-u-z}^2-2z^2$ *(by Queſtion* 102.)

Moreover, the ſum $(x+y)$ of the two terms adjacent to the middle one being $=u$, and their rectangle $=z^2$, the ſum of their ſquares $(x^2+y^2)$ will therefore be $=u^2-2z^2$ *(by the ſame.)*

And ſo, by ſubſtituting theſe values above, we get $\overline{a-u-z}^2-2z^2+u^2-2z^2+z^2=b$,

and $\frac{u^2-2z^2}{z}=a-u-z$.



Digitized by Google

Whence $a^2-2au-2az+2u^2+2uz-2z^2=b$,
and $az-u^2-uz+z^2=0$.
The double of which laſt equation, added to the former, gives $a^2-2au=b$; whence $u=\frac{a}{2}-\frac{b}{2a}$.
From which, and the equation $az-u^2-uz+z^2=0$, the value of $z$ will alſo become known.

## QUESTION CVI.

*The Sum* (a) *and the Sum of the Cubes* (b) *of five Numbers, in continued Geometrical Proportion being given; to find the Numbers.*

Retaining the notation of the laſt problem, and proceeding in the ſame manner, we have
$\overline{a-u-z}^3-\overline{3a-3u-3z}\times z^2+u^3-3uz^2+z^3=b$;
and $az-u^2-uz+z^2=0$ *(as before.)*
The firſt of which equations, by reduction, becomes
$a^3-3a^2\times\overline{u+z}+3a\times\overline{u^2+2uz}-3u^2z-3uz^2+3z^3=b$;
From whence the other equation, multiplied by $3z$, being ſubtracted, there remains
$a^3-3a^2\times\overline{u+z}+3a\times\overline{u^2+2uz}-z^2=b$;
therefore $u^2+2uz-z^2-a\times\overline{u+z}=\frac{b}{3a}-\frac{a^2}{3}$.
But, by the ſecond equation, $u^2+2uz-z^2=az+uz$;
whence, by ſubſtitution, $az+uz-a\times\overline{u+z}=\frac{b}{3a}-\frac{a^2}{3}$;
that is, $uz-az=\frac{b}{3a}-\frac{a^2}{3}$;
or $az-uz=d$ (by putting $\frac{aa}{3}-\frac{b}{3a}=d$).
From which, the ſecond equation being ſubtracted, there reſults
$u^2-z^2=d$: wherein let $\left(\frac{az-d}{z}\right)$ the value of $u$ (found from the former equation) be now ſubſtituted, and

Digitized by Google

and we shall have $\frac{\overline{az-d}^2}{z^2}-z^2=d$; and consequently

$z^4-\overline{aa-d}\times z^2+2adz-d^2=0$.

Whence $z$ will be found.

## QUESTION CVII.

*The Sum* (a), *the Sum of the Squares* (b), *and the Sum of the Cubes* (c), *of any three Numbers being given; to determine the Numbers.*

Let them be denoted by $x$, $y$ and $z$; then

since $\left\{\begin{array}{l} x+y+z=a \\ x^2+y^2+z^2=b \\ x^3+y^3+z^3=c \end{array}\right\}$ we shall, by transposition, have

$x+y=a-z$

$x^2+y^2=b-z^2$

$x^3+y^3=c-z^3$.

Now, by multiplying together the two first of these equations,

we have $x^3+x^2y+xy^2+y^3=ab-bz-az^2+z^3$.

And, by cubing of the first, we also have

$x^3+3x^2y+3xy^2+y^3=a^3-3a^2z+3az^2-z^3$: which, deducted from the treble of the former,

leaves $2x^3+2y^3=3ab-a^3+3a^2z-3bz-6az^2+4z^3$:

and, this being $=2c-2z^3$ (by the third equation) we therefore have

$6z^3-6az^2+\overline{3a^2-3b}\times z=a^3-3ab+2c$,

and consequently $z^3-az^2+\frac{a^2-b}{2}\times z=\frac{a^3-3ab+2c}{6}$.

From whence (when $a$, $b$ and $c$ are expressed in numbers) three different roots, or values of $z$, may be found, answering all the conditions of the problem.

Thus, for example, let $a=9$, $b=29$, and $c=99$; then our equation will become $z^3-9z^2+26z-24=0$.

And (by either of the two first methods explained in Sect. 12. of my *Treatise of Algebra*) the three roots, in this case, will be found to be 2, 3 and 4.

which

Which numbers are, therefore, the true values of $x$, $y$ and $z$, in the equations $x+y+z=9$, $x^2+y^2+z^2=29$, and $x^3+y^3+z^3=99$.

## QUESTION CVIII.

*The Sum* (a), *the Sum of the Squares* (b), *the Sum of the Cubes* (c), *and the Sum of the Biquadrates* (d), *of any four Numbers being given; to determine the Numbers.*

Let the four Numbers be denoted by $x$, $y$, $z$ and $u$; and put $A=a-u$, $B=b-u^2$, $C=c-u^3$, and $D=d-u^4$. From whence, by the conditions of the problem,

$x+y+z=A$,
$x^2+y^2+z^2=B$,
$x^3+y^3+z^3=C$,
$x^4+y^4+z^4=D$.

Now, if the Second of theſe equations, be ſubtracted from the ſquare of the firſt,
we ſhall have $2xy+2xz+2yz=A^2-B$.
And if, in like manner, the fourth equation be ſubtracted from the ſquare of the ſecond, we ſhall have $2x^2y^2+2x^2z^2+2y^2z^2=B^2-D$.
Moreover, if from the ſquare of the former of theſe laſt equations, the double of the latter be deducted, there will come out
$8x^2yz+8y^2xz+8z^2xy=A^4-2A^2B-B^2+2D$;
or, $8xyz\times x+y+z=A^4-2A^2B-B^2+2D$;
whence $xyz=\frac{A^4-2A^2B-B^2+2D}{8A}$ (becauſe $x+y+z=A$.)

Again, by multiplying the firſt and fifth equations into each other, we get $2x^2y+2x^2z+2y^2x+2y^2z+2z^2x+2z^2y+6xyz=A^3-AB$.
And, by multiplying the firſt and third together, there ariſes $x^3+y^3+z^3+x^2y+x^2z+y^2x+y^2z+z^2x+z^2y=AB$.

The

Digitized by Google

The double of which laſt taken from the precedent, leaves $6xyz-2x^3-2y^3-2z^3=A^3-3AB$. And this, added to, $2x^3+2y^3+2z^3=2C$,
gives $6xyz=A^3-3AB+2C$.
Hence $\frac{A^3-3AB+2C}{6}\ (=xyz)=\frac{A^4-2A^2B-B^2+2D}{8A}$
*(p. above.)*
and conſequently, by reduction, $A^4-6A^2B+8AC+3B^2-6D=0$.
In which equation let the ſeveral values of A, B, C and D, be now ſubſtituted, and then (dividing the whole by 24) we, at length, have $u^4-au^3+\frac{a^2-b}{2}\times u^2-\frac{a^3-3ab+2c}{6}\times u+\frac{a^4-6a^2b+8ac+3b^2-6d}{24}=0$: whoſe four roots (found by any of the known methods) anſwer all the conditions of the problem.

## QUESTION CIX.

*To find the leaſt Whole Number, which being divided by* 19, *ſhall produce a Remainder of* 7; *but, being divided by* 28, *the Remainder ſhall be* 13.

Let $19x+7$, denote the number ſought; where $x$, according to the queſtion, muſt be a whole number. And, by the queſtion, it likewiſe appears that $19x+7-13$, or, its equal, $19x-6$, muſt be diviſible by 28 (without a remainder.)

But it is plain that $28x$ is diviſible by 28: therefore $(9x+6)$ the difference between $19x-6$ and $28x$, muſt alſo be diviſible by the ſame number 28. For it is well known that, whatever number, or quantity, meaſures the whole, and one part, of another (without a remainder) muſt do the ſame by the remaining part. Hence $(18x+12)$ the double of $9x+6$, being diviſible by 28, if the ſame be ſubtracted from $19x-6$ (in order to get $x$ without a coefficent) the remainder, $x-18$, will, *ſtill*, be diviſible by the ſame number; and conſequently

Digitized by Google

ſequently $x-18$, either, equal to nothing, or to ſome multiple of 28. But, as the leaſt value of $x$ is required, $x-18$ muſt be $=0$. And therefore $19x+7=349$, the number required.

## QUESTION CX.

*A certain Perſon bought as many Geeſe and Ducks, together, as coſt him* 14 *Shillings; for the Geeſe he paid* 2*s.* 2*d. a-piece; and for the Ducks* 1*s.* 3*d. What Number had he of each?*

Let $x$ denote the number of the geeſe, and $y$ that of the ducks;
ſo ſhall $26x+15y=168$, *by the queſtion.*
and therefore $y=\frac{168-26x}{15}=11-x-\frac{11x-3}{15}$.
Which being a whole number, by the nature of the problem, $11x-3$ muſt, therefore, be (exactly) diviſible by 15.
But it is plain that $15x$ is diviſible by 15; and that its exceſs above $11x-3$, which is $4x+3$, muſt, *likewiſe*, be diviſible by the ſame number. Let the laſt expreſſion $(4x+3)$ be now multiplied by 3, and the preceding one $(11x-3)$ ſubtracted from the product (in order to get $x$ without a coefficent) whence you will have $x+12$; which being, *ſtill*, diviſible by 15, it is plain that $x$ muſt either be 3, or 3 added to ſome multiple of 15, as 18, 33, 48, *&c.* But it is apparent, from the nature of the queſtion, that all theſe numbers, except the firſt, are too large. Therefore there were 3 geeſe, and 6 ducks; which laſt number (the value of $x$ being known) is found directly from the equation exhibited above.

QUESTION

Digitized by Google

## QUESTION CXI.

*One having, at Play, won a certain Number of Guineas, not exceeding* 100, *and being aſked to tell the Number, made this Reply:* "*If the Number of Guineas I have* "*won be divided by* 9, *there will remain* 6; *but, if* "*the Number of Shillings contained in them be divided* "*by* 39, *there will remain* 12." *The Queſtion is, to find what Number of Guineas he was a Winner of.*

Let $9x+6$ denote the number ſought; where, according to the queſtion, $x$ muſt be a whole number. Then, the number of ſhillings being $189x+54$, it alſo appears that $\frac{189x+126-12}{39}$, or its equal $4x+2+\frac{11x+12}{13}$ muſt be a whole number; and therefore $11x+12$, diviſible by 13.

Let the number 12 (for the ſake of brevity) be denoted by $n$:

then, $13x$, and $11x+n$ being, both, diviſible by 13, their difference $2x-n$ muſt alſo be diviſible by the ſame number; and ſo, likewiſe, $\overline{2x-n}\times 5$, or its equal $10x-5n$. And, if this be ſubtracted from $11x+n$, the remainder $x+6n$ (or $x+72$) will, *ſtill*, be diviſible by 13. But $\frac{72}{13}$ is $=5+\frac{7}{13}$: therefore $x+7$ muſt be diviſible by 13; and conſequently the value of $x$, either, equal to 6, or 6 added to ſome *multiple* of 13: but, as the value of $9x+6$ is not to exceed 100 (by the queſtion) that of $x$ cannnot be greater than 6. And therefore the number ſought can be no other than 60.



Digitized by Google

## QUESTION CXII.

*A Perſon, in exchange for a Number of Pieces of Foreign-Gold, valued at 17s. 4d. each, received a certain Number of Guineas (not exceeding 50) and one Shilling over. What was the Sum exchanged?*

If $x$ be put for the number of pieces of foreign gold, and $y$ for the number of guineas; then it is plain, from the queſtion, that $52x=63y+3$; and conſequently that

$$x=\frac{63y+3}{52}=y+\frac{11y+3}{52}.$$

Here $11y+n$ (ſuppoſing $n=3$) muſt be diviſible by 52; as muſt, alſo, its quintuple $55y+5n$. And, if from this laſt, $52y$ be ſubtracted, and the remainder be multiplied by 4, we ſhall have $12y+20n$; which muſt be, ſtill, diviſible by the ſame number; and ſo likewiſe its exceſs $(y+19n)$ above $11y+n$. But $\frac{19n}{52}=\frac{57}{52}=1+\frac{5}{52}$; therefore $y+5$ is, either, equal to 52, or to ſome multiple of it; and conſequently $y$ equal, either, to 47, 99, 151, &c.

But, as the value of $y$, by the queſtion, is not greater than 50, all the numbers, after the firſt, are too large. Hence it appears that he received 47 guineas and one ſhilling, in exchange for 57 foreign-pieces; amounting in value to 49*l.* 8*s.* ſterling.

## QUESTION CXIII.

*One laid out 10 Shillings in 20 Fowls, of three different Sorts, viz. Chickens, Pigeons, and Larks: the Chickens coſt him 12d, the Pigeons 4d, and the Larks 1d, a-piece. How many had he of each?*

Let $x$, $y$, and $z$ denote the numbers of the three ſeveral ſorts, reſpectively.

Then

Digitized by Google

Then will $\left\{\begin{matrix} x+y+z=20 \\ 12x+4y+z=120 \end{matrix}\right\}$ *by the queſtion.*

And, by ſubtracting the former of theſe equations from the latter, we have $11x+3y=100$; and therefore $y=\frac{100-11x}{3}=33-3x-\frac{2x-1}{3}$. Now, $2x-1$ being diviſible by 3, it is evident that $(x+1)$ its difference from $3x$, muſt likewiſe be diviſible by 3; and, conſequently, that $x$ muſt either be 2, or 2 increaſed by ſome multiple of 3; that is, equal to ſome one of the numbers 2, 5, 8, 11, 14, *&c.* But, as neither $y$ nor $z$ can be greater than 18 (by the queſtion) ſo all the foregoing numbers, below and above 8, give the value of $y$ either too great, or too ſmall.

But, when $x$ is taken 8, $y$ will come out $=4$, and $z=8$; which are the three numbers required.

## QUESTION CXIV.

*To determine all the ſeveral Ways whereby it is poſſible to pay* 60*l. in Guineas and Moidores, only.*

Let $x$ denote the number of guineas, and $y$ the number of moidores.

Then will $21x+27y=1200$; or $7x+9y=400$, *by the problem*; and therefore $x=\frac{400-9y}{7}=57-y-\frac{2y-1}{7}$. From whence, as $2y-1$ is diviſible by 7, it will appear, by reaſoning as in the preceding examples, that $y+3$ muſt be diviſible by the ſame number; and conſequently that the leaſt value of $y$ is $=4$.

and the correſponding value of $x\left(=57-y-\frac{2y-1}{7}\right)=52$.

Now, having found the leaſt value of $y$, and the greateſt of $x$, the reſt of the anſwers will be obtained, by adding 7 (the coefficient of $x$ in the above equation) to the laſt value of $y$, *continually*, and ſubtracting 9 (the coefficent of $y$) from the laſt value of $x$. By



Digitized by Google

means of which we get the 6 following ſolutions, being all the queſtion admits of.

*viz.* $\begin{cases} x = 52,\ 43,\ 34,\ 25,\ 16,\ \text{or},\ 7, \\ y = 4,\ 11,\ 18,\ 25,\ 32,\ \text{or},\ 39. \end{cases}$

## QUESTION CXV.

*To find how many Ways it is poſſible to pay 20l. in Half-Guineas, and Half-Crowns, without any other Sort of Coin.*

If $x$ be the number of half-guineas, and $y$ the number of half-crowns; we ſhall have $21x+5y=800$; and therefore $y=\frac{800-21x}{5}=160-4x-\frac{x}{5}$. Whence it appears, at one view, that $x$ is a multiple of 5: and therefore, the ſeveral, required, values of $x$ being expreſſed by 5, 10, 15, 20, 25, 30 and 35, thoſe of $y$, anſwering thereto, muſt be 139, 118, 97, 76, 55, 34, and 13, reſpectively. So that there are 7 anſwers in this caſe.

## QUESTION CXVI.

*A Reckoning of 20 Shillings was ſpent by a Company of twenty Perſons, conſiſting of Officers, Sailors, and Marines: each Officer paid 2s. 6d. each Sailor 12d, and each Marine 8d. How many Perſons were there of each Denomination?*

Let the three required numbers be denoted by $x$, $y$, and $z$, reſpectively;

ſo ſhall $\left\{\begin{matrix} x+y+z=20 \\ 30x+12y+8z=240 \end{matrix}\right\}$ *by the queſtion.*

And, by ſubtracting 8 times the former of theſe equations from the latter, we ſhall have $22x+4y=80$; and therefore $y=20-\frac{11x}{2}$.

But, $y$ being a whole number, it is plain that $x$ muſt be an even number, and, alſo, leſs than 4; and therefore

more can be no other than 2. From whence $y$ is given $=9$, and $z=9$, likewise.

## QUESTION CXVII.

*To find a Number, which, being divided by* 28, *ſhall produce a Remainder of* 19; *but, being divided by* 19, *the Remainder ſhall be* 15; *and, being divided by* 15, *the Remainder ſhall be* 11.

Let $28x+19$ denote the number ſought; where $x$, according to the firſt condition of the problem, muſt be a whole number. And, by the ſecond condition, it appears that $28x+19-15$ muſt be diviſible by 19. Whence (following the method obſerved in the preceding examples) the leaſt value of $x$ is found $=8$: and ſo $19z+8$ (where $z$ denotes any whole number) is a general value of $x$, anſwering the two firſt conditions.

Let this be, therefore, ſubſtituted inſtead of $x$; and our aſſumed expreſſion will become $532z+243$. From whence, as $532z+232$ is diviſible by 15, the leaſt value of $z$ will be found $=14$. And $15n+14$, will be a general value of $z$: which, ſubſtituted in $532z+243$, gives $7980n+7691$ for a general anſwer to the problem; where $n$ may be, either, equal to nothing, or any whole number.

## QUESTION CXVIII.

*To find three Numbers, in the Proportion of* 5, 7, *and* 9; *which being, ſeverally, divided by* 11, 13, *and* 15, *there ſhall remain* 1, 2, *and* 3, *reſpectively.*

Let $5x$, $7x$, and $9x$ denote the three required numbers: then, by the queſtion, $5x-1$, $7x-2$, and $9x-3$, muſt be, reſpectively, diviſible by 11, 13, and 15 (without leaving any remainder.)

But it will be found (by proceeding as in the foregoing problems) that the leaſt value of $x$ to anſwer the firſt of theſe conditions, will be $=9$: therefore $9+11z$ (where

(where $z$ denotes any whole number) will be a general value of $x$, anſwering the ſame condition.

Let this value be, therefore, ſubſtituted in the ſecond and third expreſſions; which, by that means, will become $77z+61$, and $99z+78$. And then, as the former of theſe is diviſible by 13, the leaſt value of $z$, to fulfil this condition, will (alſo) be found $=9$.

Let, therefore, $9+13u$ (which is a general value for $z$) be ſubſtituted in the laſt of the three expreſſions, and it will become $\overline{13u+9}\times 99+78$. Which being diviſible by 15, the $\frac{1}{3}$ part thereof, or $\overline{13u+9}\times 33+26$ $(=429u+323)$ muſt, conſequently, be diviſible by 5. Whence $u$ is found $=3$: therefore $z$ $(=13u+9)=48$, and $x$ $(=11z+9)=537$. So that the three leaſt numbers, anſwering the conditions of the problem, are 2685, 3759, and 4833.

## QUESTION CXIX.

*Suppoſing* $6x+7y+8z=100$; *it is required to find all the poſſible values of* x, y, *and* z, *in whole Numbers.*

In queſtions of this kind, where you have three, or more, indeterminate quantities, and but one equation, it will be proper, firſt of all, to find the limits of thoſe quantities. Thus, becauſe $x=\frac{100-7y-8z}{6}=16-y-z-\frac{y+2z-4}{6}$, it appears that $x$ cannot be greater than 14. And, in the ſame manner, it will appear that $y$ cannot be greater than 12; nor $z$ greater than 10.

Now, as $x$ is a whole number, by the queſtion, $y+2z-4$ muſt therefore be diviſible by 6: and, as $2z$ and 4 are even numbers, it is plain that $y$ muſt alſo be an even number (ſince an odd one cannot be divided by an even one, without a remainder). Let $y$ be, therefore, firſt expounded by the leaſt even number (2), ſo will $y+2z-4$ become $=2z-2$: which, being diviſible by 6, it is plain that $z-1$ (the half thereof) is

diviſible

Digitized by Google

divisible by 3; and consequently that the several values of $z$ (when $y=2$) are 1, 4, 7, and 10.
Whence the corresponding values of $x$, by substituting above, will appear to be 13, 9, 5, and 1.

Let $y$ be now taken $=4$; then $y+2z-4$ will be $=2z$: and so, $z$ being divisible by 3, the several values of $z$, in this case, will be 3, 6, 9. But the two first of these, only, are for our purpose, the last giving $x=0$.

By taking $y=6$, and proceeding in the same manner, we shall get two other answers; wherein $z$ will be 2, and 5; and $x$, 7 and 3. And, by taking $y=8$, two more answers will be found (making 10 in the whole) which are all the question admits of; and which, being placed in order, will stand as below.

| $y$ | $z$ | $x$ |
|---|---|---|
| 2 | 1.4.7.10 | 13.9.5.1 |
| 4 | 3.6. | 8.4. |
| 6 | 2.5. | 7.3. |
| 8 | 1.4. | 6.2. |

## QUESTION CXX.

*If* $17x+19y+21z=400$; *it is proposed to find all the possible Values of* x, y *and* z, *in whole positive Numbers.*

When the coefficients of the indeterminate quantities $x$, $y$ and $z$ are nearly equal, as in this example, it will be convenient to substitute for the sum of those quantities: thus, let $x+y+z=m$; then, by subtracting 17 times this last equation from the preceding one, we shall have $2y+4z=400-17m$; and by subtracting the given equation from 21 times the assumed one $x+y+z=m$, there will remain $4x+2y=21m-400$. Therefore, since $y$ and $z$ can have no values less than unity, it is plain, from the first of these two equations, that $400-17m$ cannot be less than 6, and therefore $m$ not

not greater than $\frac{400-6}{17}$ or 23: alſo, becauſe by the ſecond of the two laſt equations, $21m-400$ cannot be leſs than 6, it is obvious that $m$ cannot be leſs than $\frac{400+6}{21}$ or 19: therefore 19 and 23 are the limits of $m$ in this caſe. Theſe being determined, let $4x$ be tranſpoſed in the laſt equation, and the whole divided by 2, and we ſhall have $y=10m-200-2x+\frac{m}{2}$; which being a whole number, by the queſtion, it is evident that $\frac{m}{2}$ muſt likewiſe be a whole number, and conſequently $m$ equal to an even number; which, as the limits of $m$ are 19 and 23, can only be 20, or 22: Let, therefore, $m$ be firſt taken $=20$, then $y$ will become $=10-2x$ and $z\ (m-x-y)\ 10+x$; wherein $x$ being taken equal to 1, 2, 3 and 4 ſucceſſively, we ſhall have $y$ equal to 8, 6, 4, 2, and $z$ equal to 11, 12, 13, 14, reſpectively; which are four of the anſwers required. Again, let $m$ be taken $=22$, then will $y=31-2x$ and $z=x-9$, in which let $x$ be taken equal to 10, 11, 12, 13, 14 and 15 ſucceſſively, and $y$ will come out $=11$, 9, 7, 5, 3 and 1, and $z=1$, 2, 3, 4, 5 and 6, reſpectively. Therefore we have the ten following anſwers in whole numbers; which are all the queſtion admits of.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x=$ | 1 | 2 | 3 | 4 | 10 | 11 | 12 | 13 | 14 | 15 |
| $y=$ | 8 | 6 | 4 | 2 | 11 | 9 | 7 | 5 | 3 | 1 |
| $z=$ | 11 | 12 | 13 | 14 | 1 | 2 | 3 | 4 | 5 | 6. |

QUESTION

Digitized by Google

## QUESTION CXXI.

*To find two Whole Numbers, whereof the Difference of the Squares shall be* 77.

Let the leſſer number be $x$, and the greater $x+m$; and ſuppoſe the number given to be repreſented by $a$: ſo ſhall $\overline{x+m}^2-x^2=a$,
that is, $2mx+mm=a$:
and conſequently $x=\frac{a-mm}{2m}=\frac{a}{2m}-\frac{m}{2}$.
Whence we alſo have $x+m=\frac{a}{2m}+\frac{m}{2}\left(=\frac{a+mm}{2m}\right)$.

But, in the caſe propoſed, $a$ being $=77$,
$x$ becomes $=\frac{77}{2m}-\frac{m}{2}$, and $x+m=\frac{77}{2m}+\frac{m}{2}$:
which being both required in whole numbers, it is evident, in the firſt place, that $m$ muſt be ſome diviſor of 77; and, ſecondly, that $\frac{77}{m}$ muſt be greater than $m$; and conſequently $m$ leſs than 9.

But the diviſors of 77, below 9, are 1 and 7: which numbers being wrote ſucceſſively, in the room of $m$, the correſponding values of $x$ will come out 33, and 2; and thoſe of $x+m$, 34 and 9, reſpectively. So that the queſtion, in the caſe propoſed, admits of two anſwers, and no more.

## QUESTION CXXII.

*To find a Whole Number, to which* 12 *and* 25 *being, ſucceſſively, added, both the Sums ſhall be ſquare Numbers.*

Let $z$ be the number ſought; and aſſume $x$ and $x+m$ for the roots of the two ſquares:
then will $\left\{\begin{array}{l} z+12=x^2 \\ z+25=\overline{x+m}^2 \end{array}\right\}$ *by the queſtion.*



Digitized by Google

Hence, by subtracting the former equation from the latter, we get $13=\overline{x+m}^2-x^2=2mx+mm$;
and therefore $x=\frac{13}{2m}-\frac{m}{2}$. Which being a whole number, $m$ muſt be $=1$: whence $x=6$; and $z$ $(=x^2-12)=24$.

## QUESTION CXXIII.

*To find three Whole Numbers, ſo that the Sum of the Squares of the two leaſt of them ſhall be equal to the Square of the Greateſt.*

It appears from the problem preceding the laſt, that the difference of the ſquares of $\frac{a+mm}{2m}$, and $\frac{a-mm}{2m}$ is, univerſally, equal to $a$,
or $\frac{\overline{a+mm}^2}{4mm}-\frac{\overline{a-mm}^2}{4mm}=a$; let $a$ and $m$ be what they will.
Whence it is alſo plain that $\overline{a+mm}^2=\overline{a-mm}^2+4mma$.

But, ſince it is required to have $4mma$, a ſquare number (as well as the other two) $a$ muſt therefore be a ſquare number; let it be $n^2$, and then our equation will become $\overline{nn+mm}^2=\overline{nn-mm}^2+\overline{2mn}^2$: where $m$ and $n$ may be expounded by any whole numbers, at pleaſure.

Thus, for example, ſuppoſe $m=1$, and $n=2$; then there will come out $5^2=3^2+4^2$. Again, let $m=2$, and $n=3$, and there ariſes $\overline{13}^2=\overline{5}^2+\overline{12}^2$. Laſtly, let $m=2$ and $n=5$, and you will get $\overline{29}^2=\overline{21}^2+\overline{20}^2$.

QUESTION

Digitized by Google

## QUESTION CXXIV.

*To find three Whole Numbers, whose Squares are in Arithmetical Progression.*

Let $x$, $x+m$, and $x+n$ express three such numbers. So shall $\overline{x+m}^2-x^2=\overline{x+n}^2-\overline{x+m}^2$, by the nature of the problem.

Whence $x$ is found $=\frac{nn-2mm}{4m-2n}$.

Put $4m-2n=a$, and $n^2-2m^2=b$; then $x=\frac{b}{a}$, $x+m=\frac{b+am}{a}$, and $x+n=\frac{b+an}{a}$. Now, as the squares of $\frac{b}{a}$, $\frac{b+am}{a}$, and $\frac{b+an}{a}$ are in arithmetical progression, it is plain that the squares of their *equimultiples*, $b$, $b+am$, and $b+an$, must be in arithmetical progression likewise. From whence, by expounding $m$ and $n$ by different whole numbers, successively, as many particular answers as you please, may be exhibited.

Thus, if $m=2$ and $n=3$; then, $a$ being $=2$, and $b=1$, there will come out 1, 5, and 7. But, if $m=3$ and $n=5$, we shall get 7, 13, and 17, for another answer.

## QUESTION CXXV.

*Supposing* $x^2=z^2+az+b$ *(where* a *and* b *denote given Numbers); it is required to find the Values of* x *and* z *(if possible) in Whole Numbers.*

Put $x=z+m$; then, by substitution.

$z^2+2mz+m^2=z^2+az+b$;

and consequently $z=\frac{m^2-b}{a-2m}$. Which value, by putting $a-2m=n$ (or $m=\frac{a-n}{2}$) becomes $=\frac{a^2-2an+n^2-4b}{4n}=$



Digitized by Google

$\frac{1}{4}$ of $\frac{aa-4b}{n}-2a+n$. From which it is evident, that, to have the value of $z$ a whole number, $n$ must be some divisor of the given quantity $aa-4b$, and therefore even or odd, according as $a$ is even or odd.

*Example.* Suppose the given equation to become $x^2=z^2+20z$; in which case, $a$ being $=20$, and $b=0$, we have $z=\frac{1}{4}\times\frac{400}{n}-40+n$; where the, even, divisors of of 400 are 2, 4, 8, 10, &c. whereof the second will be found to answer; the values of $z$ and $x$ coming out 16 and 24, respectively.

Again, suppose the given equation to become $x^2=z^2+100z+1000$: here we have $z=\frac{1}{4}\times\frac{6000}{n}-200+n$: And the, even, divisors of 6000, are 2, 4, 6, 8, 10, 12, 16, 20, &c. Whereof 4, 12, and 20 succeed. By the last of these (which determines the least values) $z$ comes out $=30$, and $x=70$.

## QUESTION CXXVI.

*Having given* $x^2=a^2+bz+cz^2$, *wherein* a, b *and* c *denote given Whole Numbers; it is required to find the Values of* x *and* z *(if possible) in Whole Numbers.*

Put $x=a+mz$; then will $\overline{a+mz}^2=a^2+bz+cz^2$; that is, $a^2+2amz+m^2z^2=a^2+bz+cz^2$.

Whence $z$ comes out $=\frac{b-2am}{mm-c}$:

where it is evident, that, in order to have a positive value, $m$ must be taken equal to some number between $\sqrt{c}$, and $\frac{b}{2a}$.

Thus, supposing the given equation to become $x^2=64-12z+5z^2$, the value of $m$, in this case, must be less

leſs than $\sqrt{5}$, and greater than $-\frac{12}{16}$. Let it therefore be expounded by 2 and 1, ſucceſſively; whence $\frac{-12-16m}{mm-5}$, or its equal $\frac{12+16m}{5-mm}$ (which is, here, the value of $z$) will come out 44 and 7 reſpectively; and the correſponding values of $x$ $(a+mz)$ are found to be 96 and 15: both which anſwer the conditions of the problem.

PART

# PART II.

## CONTAINING

*A Variety of* GEOMETRICAL PROBLEMS, *with their* SOLUTIONS: *both by* ALGEBRA, *and independent of it, from Principles purely* GEOMETRICAL.

---

ALTHOUGH the chief design of this work consists not in exhibiting a number of rules and precepts, but in illustrating those already known, on the subject, by a set of proper and useful examples; yet, as we are in this part to treat of the investigation of Geometrical Problems, without calling in the assistance of Algebra (a thing hitherto very little considered by authors, though in itself very interesting and useful) it may not be amiss, before we proceed to particular cases, to premise a few general observations on this head.

1. In the first place, then, it is necessary, in order to the construction of Geometrical Problems, that something of the *Geometric Loci* should be understood.—Thus it will be of use to know that the place of the vertex of a triangle, whereof the base and opposite angle are supposed to remain constant, while the other sides and angles vary, will *always* fall in the circumference of a circle passing through the extremities of the base. This appears from *Euclid*, *B.* 3. *Prop.* 21; and is also demonstrated in my *Elements of Plane Geometry*, *B.* 3, *Prop.* 9, 1*st edit.* or *B.* 3, *Theor.* 11, 4*th edit.*

It ought moreover to be known, that the locus of the vertex, when the ratio of the two sides and the base

Digitized by Google

base of the triangle are given, or continue invariable, will, *also*, be the circumference of a circle, dividing the base in the given ratio. *(For the demonstration of which, see Elem. Plane Geometry, B. 4, Prop. 15, 1st edit.)*

If the sum, or the difference, of the squares of the two sides, together with the base, be supposed given, the locus of the vertex will, *still*, be the circumference of a circle, in the former case; and a right-line, in the latter *(vid. B. 2, Prop. 12, 1st edit.* or *B. 2, Theor. 11, 4th edit.*

But, if the sum, or the difference, of the sides, themselves, is given; then the locus will, either, be the arch of an ellipsis or an hyperbola (as is well known to those who have touched upon conic-sections.) But these two last are not admitted in the construction of linear and plane problems; of which, only, we purpose to treat.——The problems, of the following collection, wherein the use of the *Geometric-Loci*, above specified, is particularly exemplified, are the 11, 12, 19, 34, 35, 48, and 52d.

2° When, in the figure to be constructed, the sum, or the difference, of two adjacent sides happens to be given; it will be proper, first, to form a triangle, so that the said sum, or difference, may be one of its sides; and, then, to consider, what other sides, or angles, will be given, or become known, in consequence thereof. This rule is illustrated in the 1, 2, 18, 29, 30th, and some other of the succeeding problems.

3° It often happens that the ratio of two, or more, lines is given, from the nature of the figure, or by hypothesis, though the lines themselves are absolutely unknown: in all such cases we must endeavour, by drawing parallels (or some other way) to obtain other lines in the same given ratio: so that, one of them being given, or known from the nature of the figure or problem, the other may also become known.—The use of this rule, which is very extensive, will particularly

larly appear in the ſolutions to the 3, 13, 16, 22, 26, 33, 37, and 56th. problems.

4°. But, if the lines, whereof the ratio is given, ſhould happen to lye remote of each other; then, by the help of thoſe, we muſt endeavour to determine the ratio of others, lying nearer together; and ſo on, till we obtain (if poſſible) the ratio of two lines, that are both ſides of the ſame triangle; wherein one angle and the remaining ſide (or ſome other, two, parts) are given.—For the better underſtanding this rule, conſult the ſolutions to the 49th and 57th problems, in particular.

5°. Laſtly, when the rectangle under two unknown lines is given, either, a mean proportional muſt be found, or elſe, two other lines muſt be aſſigned, by forming ſimilar triangles (or ſome other way) ſo as to comprehend an equal rectangle; and ſo that, one of them being given by the nature of the figure, the other may alſo become known. This rule is exemplified in the 4, 5, 6, 9, 10, 21, 38, 40, 41, 44, and 53d. problems.

Beſides the above, other obſervations might be here laid down; but thoſe already delivered being the moſt general that have occurred to me, I ſhall now proceed on in the reſolution of problems; the proper buſineſs of this work.

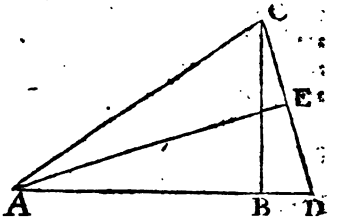
PROBLEM

Digitized by Google

## PROBLEM I.

*One Leg* BC, *and the Difference between the other Leg* AB *and the Hypothenuse* AC, *of a right-angled Triangle* ABC, *being given; to find both* AB *and* AC.

Put BC$=a$, AB$=x$, and AC$=x+b$; ($b$ being the given difference) then $\overline{AC}^2$ being $=\overline{AB}^2+\overline{BC}^2$ (*Elem.* 8. 2.*) we have $xx+2bx+bb=xx+aa$: whence $2bx=aa-bb$; and consequently $x=\frac{aa-bb}{2b}=\frac{aa}{2b}-\frac{b}{2}$. From which AC $(x+b)$ is given $=\frac{aa}{2b}+\frac{b}{2}$.




*Geometrically.*

If, in AB produced, there be taken BD equal to the given difference of AC and AB, and DC be drawn (according to the second general observation) it is evident that AD will be equal to AC; and the angle ACD, also, equal to the angle D.

Therefore, having taken BD as above specified, and made BC perpendicular thereto, and of the given length, and joined D, C; let CA be so drawn as to make the angle DCA=D; or, instead thereof, let a perpendicular EA be erected on the middle of CD; then the intersection of either of these lines with DB, produced, determines the triangle.

From this construction we have the very same theorem, for the numerical solution, as is derived above, from the algebraical process: for the triangles ADE and CDB, having each a right-angle, and D common,

* *Note. The quotations, in this, and the succeeding problems, refer to the author's Elements of Geometry, 4th edit. printed for* F. WINGRAVE.

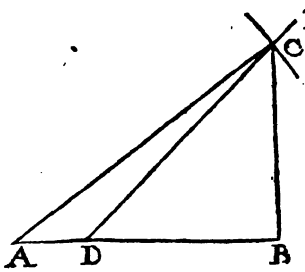


Digitized by Google

common, are similar. Therefore DB : DC : : DE ($\frac{1}{2}$DC) : AD (AC) $= \frac{DC^2}{2DB} = \frac{BC^2+BD^2}{2BD} = \frac{BC^2}{2BD} + \frac{BD}{2}$; *as before.*

## PROBLEM II.

*The Hypothenuse* AC, *and the Difference of the two Legs* AB *and* BC, *of a right-angled triangle* ABC, *being given; to determine the Legs.*




Put AC $= a$, AB $= x$, and BC $= x - b$: so shall $xx + \overline{x-b}^2 = aa$ (*Elem.* 8. 2.) that is, $2xx - 2bx + bb = aa$. Whence $xx - bx = \frac{aa}{2} - \frac{bb}{2}$: and consequently $x = \sqrt{\frac{1}{2}aa - \frac{1}{4}bb} + \frac{1}{2}b$.

*Geometrically.*

If, in AB there be taken AD equal to the given difference, and CD be drawn; then, DB being $=$BC, the angle BDC will also be $=$BCD$=$ half a right-angle.

Therefore, having laid down AD, and drawn an indefinite line DCE to make the angle BDE $=\frac{1}{2}$ right-angle; upon the center A, with the given interval AC, let an arch be described, intersecting DE in C; from which point, upon AD produced, let fall the perpendicular CB; so shall ABC be the triangle required.

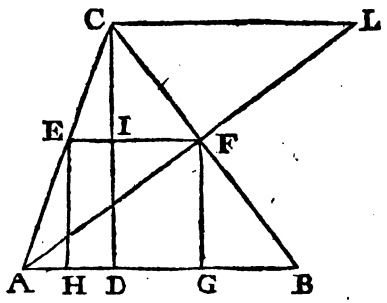
The numerical solution, according to this construction, is very easy, by the help of Trigonometry: for, two sides and one angle of the triangle ADC being given, the other angles may from thence be found; and then, all the angles and one side (AC) of the proposed triangle being known, the other sides AB and BC may also be determined.

PROBLEM

## PROBLEM III.

*The Baſe* AB *and the Perpendicular* CD *of any Triangle* ABC *being given; to find the Side* EF, *or* EH, *of the inſcribed Square* EFGH.

Put CD $=a$, AB $=b$, and DI (=EF) $=x$: then will CI $=a-x$; and, by reaſon of the ſimilar triangles ABC and ECF, we ſhall have $a : b :: a-x : x$ (= EF.) Therefore $ax = ab-bx$; and conſequently $x = \frac{ab}{a+b}$.




*Geometrically.*

The ratio of EH to EA being given, as CD to CA, EF muſt therefore be to EA in the ſame given ratio: and, if CL be drawn parallel to EF, meeting AF produced in L *(agreeable to the* 3[d] *general obſervation)* it is evident, becauſe of the ſimilar triangles, that the line CL, ſo drawn, will be to CA, *ſtill*, in the ſame given ratio; that is, CL : CA : : CD : CA; and conſequently CL=CD. Whence the method of conſtruction is manifeſt.

## PROBLEM IV.

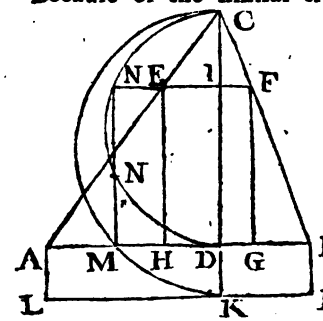
*To determine the Sides of a Rectangle.* EFGH, *inſcribed in a given Triangle* ABC, *whoſe Area ſhall be to that of the Triangle in a given Ratio.*

Put the perpendicular CD $=a$, the baſe AB $=b$, and the altitude EH of the rectangle $=x$; and let the given ratio of ABC to EFGH be that of $m$ to $n$.



Digitized by Google

Becauſe of the ſimilar triangles ABC and EFC, it will be, CD *(a)* : AB *(b)* : : CI *(a−x)* : EF = $\frac{ba-bx}{a}$ *(Elem.* 20. 4.) Therefore EF × EH = $\frac{abx-bx^2}{a}$; and conſequently $m : n : : \frac{ab}{2} : \frac{abx-bx^2}{a}$ *(by the queſtion.)*




Hence $ax - xx = \frac{naa}{2m}$; and therefore $x = \frac{a}{2} \pm \sqrt{\frac{aa}{4} - \frac{naa}{2m}}$.

*Geometrically.*

The rectangle HF being to the triangle ABC in a given ratio, and the latter of theſe being actually given, the magnitude of the former is alſo given; and therefore may be expreſſed by a given rectangle ABPL, on the baſe AB; whoſe altitude KD is to half that of the triangle in the aforeſaid given ratio.

But it appears that the rectangle DI × EF is to the rectangle DI × IC in the given ratio of EF to IC, or of AB to CD *(Elem.* 1. *and* 20. 4); and that the rectangle DK × AB is alſo to DK × CD in the ſame given ratio. Therefore, the antecedents being equal, the conſequents muſt likewiſe be equal, or DI × IC = DK × CD. Whence this conſtruction.

Deſcribe, upon CD and CK, two ſemi-circles; and, from the point M wherein the circumference of the latter cuts AB, let MN be drawn, parallel to DC, interſecting the former in N; ſo ſhall MN be the required altitude of the rectangle. Since DI × CI = $\overline{IN}^2$ = $\overline{DM}^2$ = DK × CD *(Elem.* 19. 4), *as above.*

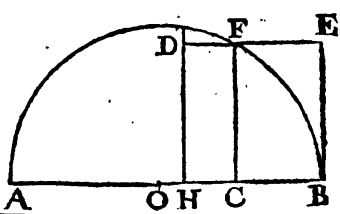
This

This problem, it is obſervable, becomes impoſſible when MN paſſes intirely without the leſſer ſemi-circle, that is, when the given rectangle is ſuppoſed greater than half the triangle. The ſame thing appears alſo from the algebraic ſolution, in which caſe the quantity $\left(\frac{aa}{4}-\frac{naa}{2m}\right)$ under the radical-ſign, becomes negative.

## PROBLEM V.

*To divide a given Right-line* AB *into two ſuch Parts* AC *and* BC, *that the Rectangle contained under them may be of a given Magnitude.*

Put AB$=a$, and AC$=x$, and let the given magnitude, or content, of the propoſed rectangle be repreſented by the ſquare BD, whoſe ſide BE, or ED, let be denoted by $b$: then will $x \times a-x = bb$; or $xx-ax=-bb$.




Whence $x=\pm\sqrt{\frac{1}{4}aa-bb}+\frac{a}{2}$.

*Geometrically.*

If upon AB, as a diameter, a ſemi-circle AFB be deſcribed, it is evident *(by Elem.* 19. 4.) that a perpendicular FC, drawn from the point wherein the circumference interſects DE, will cut AB in the point required.—It is plain, from both theſe ſolutions, that the given rectangle muſt not be greater that the ſquare of half the given line, to be divided.

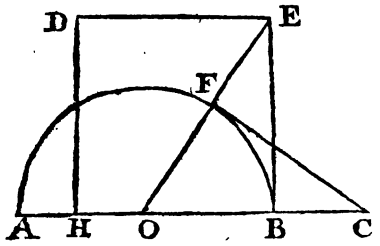
PROBLEM

Digitized by Google

## PROBLEM VI.

*To a given Line* AB, *it is required to add another Line* BC, *ſo that the Rectangle under the whole, compounded, Line* AC, *and the Part added, may be of a given Magnitude.*

Let AB=$a$, BC=$x$, and the ſide of the given ſquare BEDH, expreſſing the magnitude of the propoſed rectangle, =$b$. Then we ſhall have $\overline{a+x} \times x = bb$; and conſequently $x = \sqrt{bb+\frac{1}{4}aa}-\frac{1}{2}a$.




*Geometrically.*

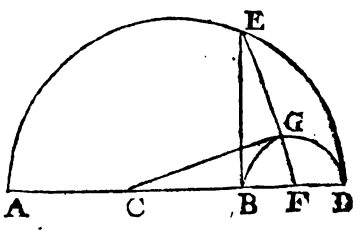
If, upon AB, a ſemi-circle be deſcribed, and CF be ſuppoſed drawn to touch it in D, it is plain, *from Elem. Corol. to* 22. 3, that $\overline{CF}^2$ is =AC×BC=$\overline{BE}^2$ *(by hypotheſis)*; and conſequently CF=BE: therefore, OF being =OB, it follows that OE and OC are likewiſe equal. Hence, if to the middle of AB, we draw EO, and take OC=EO, *the thing is done.*

PROBLEM

## PROBLEM VII.

*To divide a given Right-line* AB *into two ſuch Parts, that, the Rectangle under one of them* AC *and another, given, Line* BD, *may be equal to the Square of the remaining Part* BC.

Put AB=$a$, BD=$b$, and BC=$x$; then will AC=$a-x$, and therefore $xx=\overline{a-x}\times b$, *by the queſtion.* Hence $xx+bx=ab$; and, conſequently $x=\sqrt{ab+\frac{1}{4}bb}-\frac{1}{2}b$.




*Geometrically.*

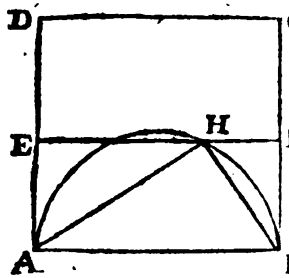
Since $\overline{BC}^2=AC\times BD$ *(by hypotheſis)* it follows, by adding BC×BD to each, that $\overline{BC}^2+BC\times BD=AC\times BD+BC\times BD$; or that, BC × CD = BD × AB = $\overline{BE}^2$, taking BE a mean proportional between BD and AB *(by Elem.* 14. 5.) But the rectangle BC×CD, if a ſemi-circle be deſcribed upon the diameter BD, is known to be equal to the ſquare of the tangent CG *(Elem. Cor. to* 22. 3). Hence $\overline{CG}^2=\overline{BE}^2$; and conſequently CG=BE: therefore, FG being alſo =FB, it follows that FC is equal to FE; whence the method of conſtruction is manifeſt.

PROBLEM

Digitized by Google

## PROBLEM VIII.

*To determine two Lines, whereof the Rectangle ſhall be equal to a given Rectangle* ABFE, *and the Sum of their Squares equal to a given Square* ABCD.




Put AB (=BC) =$a$, BF (=AE) =$b$, and let the two required lines be denoted by $x$ and $y$.

Then will $xy=ab$, and $xx+yy=aa$, *by the queſtion*: whence by adding, and ſubtracting, the double of the former of theſe equations from the latter,

we have $\begin{cases} xx+2xy+yy\ (=\overline{x+y}^2) &= aa+2ab, \\ xx-2xy+yy\ (=\overline{x-y}^2) &= aa-2ab. \end{cases}$

Therefore $\left\{\begin{matrix} x+y=\sqrt{aa+2ab} \\ x-y=\sqrt{aa-2ab} \end{matrix}\right\}$

From which $\left\{\begin{matrix} 2x=\sqrt{aa+2ab}+\sqrt{aa-2ab} \\ 2y=\sqrt{aa+2ab}-\sqrt{aa-2ab}. \end{matrix}\right\}$

*Geometrically.*

If upon AB a ſemi-circle be deſcribed, interſecting EF in H, then the lines joining A, H, and B, H will anſwer the conditions of the problem. For, the angle AHB being a right-one *(Elem.* 13. 3.) thence is $\overline{AH}^2+\overline{BH}^2=\overline{AB}^2$=ABCD *(Elem.* 8. 2.); and AH× BH (=twice the triangle ABH) =ABFE *(Elem. Corol. to* 2. 2.)

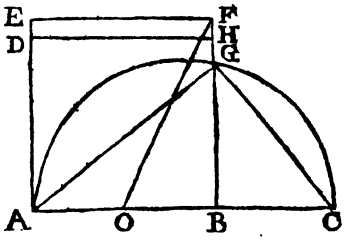
PROBLEM

Digitized by Google

## PROBLEM IX.

*To determine two Lines, whereof the Rectangle ſhall be equal to a given Rectangle* ABFE, *and the Difference of their Squares equal to a given Square* ABHD.

Put AB$=a$, BF$=b$; and let $x$ be the greater, and $y$ the leſſer of the two lines required: ſo ſhall $xy=ab$; and $xx-yy=aa$, *by the queſtion.* From the former of which equations we have $y=\frac{ab}{x}$; which




value, ſubſtituted in the latter, gives $xx-\frac{aabb}{xx}=aa$. Hence $x^4-a^2x^2=a^2b^2$; and conſequently $x=\sqrt{\ :\frac{1}{2}aa+a\sqrt{bb+\frac{1}{4}aa}\ }$: whence $y$ will alſo be known.

*Geometrically.*

It is evident, in the firſt place, that the two lines to be determined will be the hypothenuſe and one leg of a right-angled triangle (ABG) whoſe remaining leg is the given line AB. And, ſince the rectangle under theſe lines is ſuppoſed given, another triangle ACG, ſimilar to ABG, muſt therefore be aſſumed; ſo that CG, in the former, and BG, in the latter, may be homologous ſides *(according to the 5th general obſervation; vid. p.* 88.)

Hence we have CG$\times$AB$=$BG$\times$AG *(Elem.* 24. 3.) $=$AB$\times$BF *(by hyp.)* and conſequently CG$=$BF.

And ſo, AC$\times$BC $(=\overline{CG}|^2$, *Elem. Cor.* 19. 4.) being given $=\overline{BF}|^2$, the caſe under conſideration is now reduced to our 6th problem: whence we have the following conſtruction.



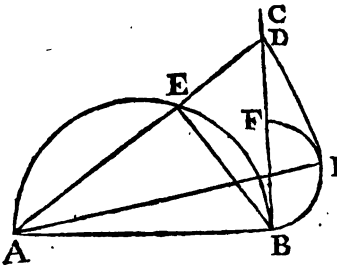
Digitized by Google

To the middle of AB, let FO be drawn; and, having taken OC=OF, let a ſemi-circle be deſcribed upon AC, cutting BF in G; ſo ſhall BG be the leſſer, and AG the greater of the two lines required.

## PROBLEM X.

*The Diameter* AB *of a Semi-circle being given, to find a Point* D *in the Perpendicular* BC, *from whence* DA *being drawn, the Part thereof,* ED, *without the Semi-circle, ſhall be of a given length* (BF.)

Put AB=$b$, DE (=BF) =$a$, and AD=$x$; alſo let BE be drawn. Becauſe the angle AEB is a right-one, the triangles ADB and ABE are ſimilar. And therefore AD ($x$) : AB ($b$) :: AB ($b$) : AE ($x-a$). Whence $xx-ax=bb$; and conſequently $x=\sqrt{bb+\frac{1}{4}aa}+\frac{1}{2}a$.




*Geometrically.*

Since DA×EA is =AB² (*Elem.* 24. 3.) where the part DE of DA is given (=BF) the caſe is therefore, reduced to our 6th problem.

From whence it will appear, that, if upon BF a ſemi-circle be deſcribed, and through the center thereof, AH be drawn, meeting the periphery in H, an arch deſcribed from the center A, with the radius AH, will cut BC in the point required.

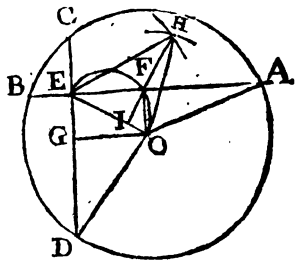
## PROBLEM XI.

*Having the Length of two Chords* AB *and* CD, *cutting each other at Right Angles, together with the Diſtance* OE *of the Point of their Interſection from the Center; to determine the Diameter of the Circle.*

Upon the given chords, from the center O, let fall the

Digitized by Google

the perpendiculars OF and OG; and draw the radii OA and OD: alſo put AF $(=\frac{1}{2}AB)=a$, DG $(=\frac{1}{2}CD)=b$, OE $=c$, and AO $(=DO)=x$. Then will $\overline{OF}|^2=x^2-a^2$, and $\overline{OG}|^2=x^2-b^2$; but $\overline{OF}|^2+\overline{OG}|^2$ $(=\overline{OF}|^2+\overline{FE}|^2)$ is $=\overline{OE}|^2$; that is, $2x^2-a^2-b^2=c^2$; and conſequently $x=\sqrt{\frac{aa+bb+cc}{2}}$.




Whence the diameter is given $=\sqrt{2a^2+2b^2+2c^2}$.

*Geometrically.*

Since $\overline{FE}|^2+\overline{DG}|^2$ is $(=\overline{OD}|^2=\overline{OA}|^2)$ $=\overline{OF}|^2+\overline{AF}|^2$, it is evident that $\overline{FE}|^2-\overline{OF}|^2$ is given $=\overline{AF}|^2-\overline{DG}|^2$. We are therefore to conſtruct a right-angled triangle upon the given hypothenuſe OE, whereof the ſquares of the two legs ſhall have the ſame difference as the two given ſquares $\overline{AF}|^2$ and $\overline{DG}|^2$.

In order to which, upon OE, let a ſemi-circle be deſcribed; alſo, from the centers O and E, with radii equal to DG and AF, reſpectively, let two arcs be deſcribed, interſecting each other in H; from which point, upon OE, let fall the perpendicular HI; which will interſect the ſemi-circle in (F) the vertex of the required triangle: ſince it is evident that $\overline{FE}|^2-\overline{OF}|^2$ $=\overline{EI}|^2-\overline{OI}|^2=\overline{EH}|^2-\overline{OH}|^2=\overline{AF}|^2-\overline{DG}|^2$ *(by conſtruction.*

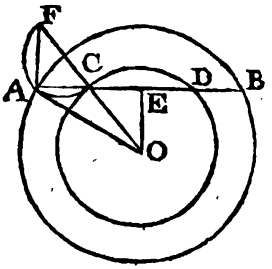
Therefore if in EF produced, FA be taken of the given length, OA (when drawn) will be the radius of the required circle.



Digitized by Google

## PROBLEM XII.

*To draw a Right-line to cut two given concentric Circles, OAB, OCD, ſo that the Chords, or Parts of the ſaid Line intercepted by thoſe Circles, may obtain a given Ratio.*




Put the radius OA of the greater circle $=a$, and the radius OC of the leſſer $=b$; and let the given ratio of AB to CD be that of $m$ to $n$: then, denoting OE, the diſtance of AB from the center by $x$, we have $AE^2=aa-xx$, and $CE^2=bb-xx$. Therefore, AE being $=\frac{1}{2}AB$, and $CE=\frac{1}{2}CD$, it follows, that, $aa-xx : bb-xx :: m^2 : n^2$; and conſequently $n^2a^2-n^2x^2=m^2b^2-m^2x^2$: whence $x=\sqrt{\frac{mmbb-nnaa}{mm-nn}}$. By means of which AB may be drawn.

*Geometrically.*

Since the ratio of AE to CE is given, let OC be produced to F, ſo that OF may be to OC in the ſame, given ratio, *(agreeable to the 3d general obſervation)* then, A, F being joined, the triangle CAF will be ſimilar to the triangle CEO; and conſequently the angle CAF a right one. Hence the following conſtruction.

Having drawn the radius OC, and in it, produced, taken OF in proportion thereto, as AB is to CD *(as above ſpecified)* let a ſemi-circle, upon CF be deſcribed, interſecting the greater of the two given circles in A; from which point, through C, draw AB, *and the thing is done.*

It is manifeſt, both from this, and the algebraical ſolution, that the ratio of $m$ to $n$ (or of AB to CD) cannot

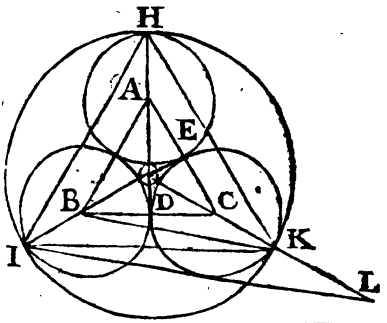
Digitized by Google

cannot be given leſs than that of OA to OC, without rendering the problem impoſſible.

## PROBLEM XIII.

*To determine the Radii of three equal Circles,* A, B, C, *deſcribed in a given Circle* HIK *to touch each other and likewiſe the Circumference of the given Circle.*

Let the centers of the ſeveral circles be joined; and let AO and BO be produced to biſect BC and AC in D and E: alſo let the radius (OI) of the given circle be denoted by $a$, and that of each of the required ones by $x$. Now the triangles BCE and BOD being ſimilar, and $CE=\frac{1}{2}BC$, it appears that OD is alſo $=\frac{1}{2}OB$. But $\overline{OB}^2-\overline{OD}^2$ is $=\overline{BD}^2$; that is, in ſpecies, $\overline{a-x}^2-\frac{\overline{a-x}^2}{4}=xx$. Which, ſolved, gives $x=\sqrt{12aa}-3a=a\times 2\sqrt{3}-3$.




*Geometrically.*

It is evident that the right-line IK, joining the points of contact I and K, is the ſide of an equilateral triangle inſcribed in the given circle: and, that, if in OK produced there be taken $KL=\frac{1}{2}IK$, a line drawn from I to L, will be parallel to another line drawn from B to K; becauſe the triangles IKL and BCK (having IKL=BCK, and IK : KL (:: 2 : 1) :: BC : CK) are equiangular.

Therefore, in order to the geometrical conſtruction, having firſt drawn the radii OH, OI, and OK, to divide

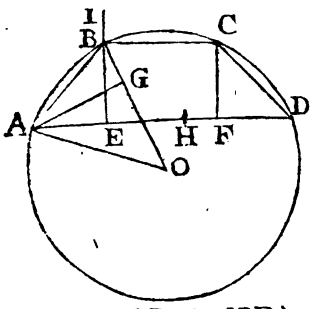
divide the circumference in three equal parts, and taken KL, in OK produced, equal to $\frac{1}{2}$IK; draw LI, and KB, parallel thereto, meeting OI in B; make OA and OC each = OB; and upon the centers A, B, and C, through A, I, K let the three required circles be deſcribed.

## PROBLEM XIV.

*Suppoſing* AB *and* ABD *to be two Arcs of a Circle, in the Ratio of* 1 *to* 3; *and that both their Chords* AB *and* AD *are given; it is required to find the Radius of the Circle.*

Let the arch BD be equally divided in C, and upon AD let fall the perpendiculars BE and CF; alſo let AG be perpendicular to the radius OB: and put AB=$a$, AD=$b$, and AO (=OB) =$x$.

It is evident, becauſe AB, BC and CD are all equal, that AE=DF, and EF=BC=$a$: therefore AE + DF = $b-a$, and AE = $\frac{b-a}{2}$. It alſo appears that the triangles ABE and AOG are equiangular, becauſe the angle BAD, ſtanding upon the arch BD, is equal to the angle O, at the center, ſtanding upon AB (=$\frac{1}{2}$BD): hence we have, AB *(a)* : AE $\left(\frac{b-a}{2}\right)$ : : AO *(x)* : OG = $\frac{\overline{b-a}\times x}{2a}$. But $\overline{AB}^2=\overline{OA}^2+\overline{OB}^2-2OB\times OG$; that is, in ſpecies, $a^2=x^2+x^2-\frac{\overline{b-a}\times x^2}{a}$; or $a^3=3ax^2-bx^2$. Therefore $x=\sqrt{\frac{a^3}{3a-b}}=a\sqrt{\frac{a}{3a-b}}$.




From

Digitized by Google

From the ſame equation, if the radius AO *(x)* and the chord *(b)* of an arch ABD be ſuppoſed given, the chord AB *(a)* of the ſub-triple of that arch, may be determined: but this, by-the-bye.

*Geometrically.*

The geometrical conſtruction of the propoſed problem is alſo obvious from the known value of AE and the equality of the angles O and EAB; and is thus. Draw AD of the given length, from which take DH =AB; let the remainder AH be biſected by the perpendicular EI; to which draw AB ſo as to be of the given length; and upon the ſame, as a baſe, let an iſoſceles triangle AOB be conſtituted, whoſe vertical angle O ſhall be =EAB; then it is evident that either of the equal ſides AO, or BO, will be the radius of the circle.

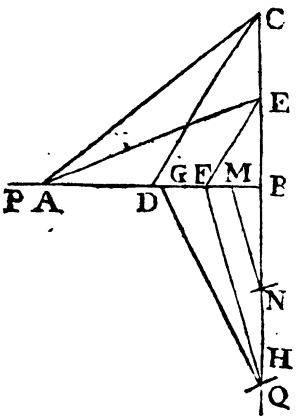
As to the trigonometrical calculation, it is too plain, from the conſtruction, to need any thing further to be ſaid about it.

PROBLEM

Digitized by Google

## PROBLEM XV.

*Having the Lengths of two Lines* AE *and* CD *drawn from the acute Angles of a right-angled Triangle* ABC, *to bisect, and terminate in, the oppoſite Sides; to determine the Triangle.*




Call AE, $a$; CD, $b$; and BD (or AD) $x$; then will $\overline{CB}^2 = b^2 - x^2$; whence $\overline{BE}^2$ $(= \frac{1}{4}\overline{CB}^2)$ $= \frac{bb - xx}{4}$; and therefore $a^2$ $(= \overline{AE}^2 =$ $\overline{BE}^2 + \overline{BA}^2)$ $= \frac{bb - xx}{4} + 4xx$.

Hence $15xx = 4aa - bb$; and $x = \sqrt{\frac{4aa - bb}{15}}$.

*Geometrically.*

If EF be ſuppoſed parallel to CD, meeting AB in F, the length thereof, being half that of CD, will conſequently be given: whence $\overline{AB}^2 - \overline{BF}^2$ $(= \overline{AE}^2 - \overline{EF}^2)$ is alſo given.

Therefore the problem is reduced to this; to determine two lines AB and BF, in the ratio of 4 to 1, ſo that the difference of their ſquares may be equal to the difference of the ſquares of two given lines AE and EF.

Hence, having drawn two indefinite lines BP and BQ at right-angles to each other, take BG, in the former, equal to EF; and from the point G, to BQ, draw

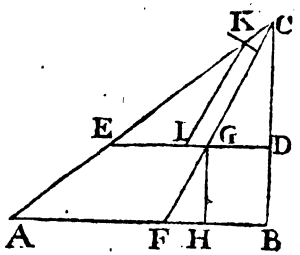
Digitized by Google

draw GH=AE: so shall $\overline{AB}^2-\overline{BF}^2$ $(=\overline{AE}^2-\overline{EF}^2)$ $=\overline{BH}^2$ *(Elem. Cor.* 8. 2). Therefore, if from any point M in PB, to BQ there be drawn MN=4MB; it is evident that a line, HF, drawn from H parallel to NM, will cut off BF as required.——This problem becomes impossible when either of the two given lines is greater than the double of the other.

## PROBLEM XVI.

*The Length and Position of a Right-line* DE, *drawn parallel to the Base of a given right-angled Triangle* ABC *being known; it is proposed to draw another Right-line* CF *from the Vertex of the Triangle, so that the Part thereof* (FG) *intercepted by* AB *and* ED, *may be equal to* (EG) *the Part of* ED *intercepted by* CA *and* CF.

Upon AB let fall the perpendicular GH: and put ED=$a$, CD=$b$, DB=$c$, and EG=$x$: then, from the similarity of the triangles CDG and GHF, it will be, CD ($b$): DG ($a-x$) :: GH ($c$) : HF $=\frac{c\times\overline{a-x}}{b}$:




and therefore $\frac{c^2\times\overline{a-x}^2}{b^2}+c^2$ $(HF^2+GH^2)=x^2$ $(GF^2.)$

Whence, by reduction, $b^2x^2-c^2x^2+2c^2ax=c^2a^2+c^2b^2$;

and $x^2+\frac{2ac^2}{bb-cc}\times x=\frac{\overline{a^2+b^2}\times c^2}{bb-cc}$.

From which $x$ is found $=\frac{bc\sqrt{aa+bb-cc}-ac^2}{bb-cc}$.

P *Geometrically.*

Digitized by Google

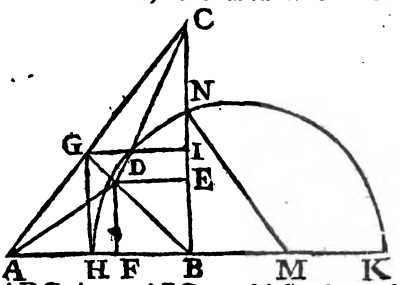
*Geometrically.*

Since GF is to GC every where in the given ratio of DB to DC, GE, in the required position, must therefore be to GC, in the same, given, ratio. Hence, if, in ED, there be taken EI=DB, and from the center I, at the distance of CD, an arch be described, intersecting EC in K; then a line CGF, drawn parallel to the radius KI, will determine both the length and position of GF: for it is evident that CG : EG :: IK : EI :: CD : DB :: CG : GF, and consequently that EG=GF.—This problem appears to be impossible when BD is greater than CE.

## PROBLEM XVII.

*Supposing the Area of a Square* BE DF, *formed within a given right-angled Triangle* ABC, *to be equal to the Area of the Triangle* ADC, *made by drawing Lines from the Extremes of the Hypothenuse to the adjacent Angle* D *of the Square; it is proposed to determine the Side of the Square.*

Call BC, $a$; BA, $b$; and BE (or BF) $x$: then, CE being $=a-x$ and AF $=b-x$, the area of the triangle CED will be $=\frac{x\times\overline{a-x}}{2}$, and that of the triangle AFD $=\frac{x\times\overline{b-x}}{2}$.




But it is evident that BEDF +CED+AFD+ADC is =ABC: which, because ADC is equal to BEDF, also gives $2BEDF+CED+AFD=ABC$, that is, $2x^2+\frac{x\times\overline{a-x}}{2}+\frac{x\times\overline{b-x}}{2}=$

$= \frac{ab}{2}$: hence $x^2 + \frac{a+b}{2} \times x = \frac{ab}{2}$; and $x = \sqrt{\frac{ab}{2} + \overline{\frac{1}{4}a + \frac{1}{4}b}^2} - \frac{a+b}{4}$. Q. E. I.

*Geometrically.*

If BD be produced to meet AC in G, and GH be drawn perpendicular to AB, it is evident that the triangle ADC will be to the triangle ABC $\left(\frac{AB \times BC}{2}\right)$ as GD : GB, or as HF to HB *(Elem. 8. and 14. of the 4)*: and it alſo appears *(by Elem. 14. 4. 1ſt edit.)* that $HB \times \overline{AB+BC}$ is $= AB \times BC$. Therefore it follows that triangle ADC : $\frac{HB \times \overline{AB+BC}}{2}$ : : HF : HB : : HF $\times \frac{AB+BC}{2}$ : $HB \times \frac{AB+BC}{2}$; and conſequently that the triangle $ADC = HF \times \frac{AB+BC}{2} = HF \times BK$; by taking BK (in AB produced) equal to $\frac{AB+BC}{2}$.

Hence, $\overline{BF}^2$ being $(=ADC) = HF \times BK$, the caſe under conſideration is reduced to our 7th problem; and the geometrical conſtruction will, therefore, be as follows.

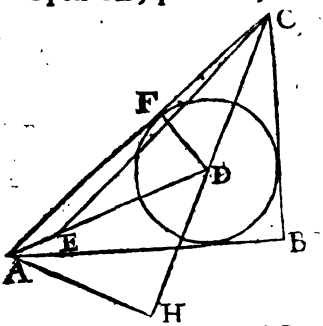
Having drawn BG (to biſect the angle ABC) and GH perpendicular to AB, and alſo taken BK equal to half the ſum of AB and BC *(as above intimated)*, let a ſemi-circle, upon HK, be next deſcribed, interſecting BC in N; from which point, to the middle of BK, let NM be drawn; then make MF = MN, and BF will be the ſide of the ſquare; as is manifeſt from the problem above quoted.



Digitized by Google

## PROBLEM XVIII.

*Having given the Hypothenuſe* AC *of a right-angled Triangle* ABC, *and the Difference of two Lines* AD *and* CD, *drawn from the Extremes thereof to the Center* D *of the inſcribed Circle; to determine the remaining Sides* AB *and* BC, *of the Triangle.*




Upon CD, produced, let fall the perpendicular AH; and make $AC=a$, $AD=x$, $DC=y$, and the given difference, $x-y=b$.

It is evident that the angle ADH is $=DAC+DCA=\frac{1}{2}BAC+\frac{1}{2}BCA=\frac{1}{2}$ a right angle; and therefore $AH=HD=\frac{AD}{\sqrt{2}}=\frac{x}{\sqrt{2}}$. But $CD^2+AD^2+2DH\times CD=AC^2$; that is, in Species, $y^2+x^2+xy\sqrt{2}=a^2$. Which equation, by ſubſtituting $x-b$ inſtead of $y$ its equal, and denoting $\sqrt{2}$ by $c$, becomes $x^2-2bx+b^2+x^2+cx^2-cbx=a^2$; that is $\overline{2+c}\times x^2-\overline{2+c}\times bx+b^2=a^2$: whence $x^2-bx=\frac{aa-bb}{2+c}$, and $x=\sqrt{\frac{aa-bb}{2+c}+\frac{1}{4}bb}+\frac{b}{2}$.

*Geometrically.*

The geometrical conſtruction of this problem, as the angle ADH is given ($=\frac{1}{2}$ a right angle) is exceeding obvious: for, if DE be ſuppoſed $=DC$, ſo that AE may expreſs the given difference of AD and CD, the angle DEC (ſuppoſing CE drawn) will be given $=\frac{1}{2}ADH$. Therefore the triangle AEC, by means of the given angle AEC, and the two given ſides AE and AC, may be conſtructed. And then, by producing

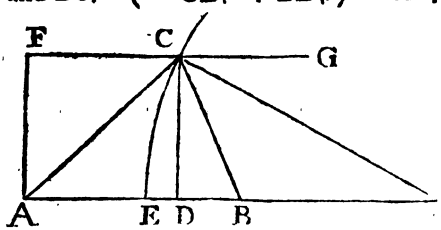
Digitized by Google

producing AE, and making the angle ECD=DEC, the point D will likewise be determined; and consequently the radius of the circle, by letting fall a perpendicular DF, upon AC: whence the circle itself may be described; and two lines may be drawn from A and C to touch the same; and thereby form the triangle ABC, *as required.*

## PROBLEM XIX.

*Having the Base* AB, *the Perpendicular* CD, *and the Ratio of the two Sides* AC, BC, *of a Triangle* ABC; *to find the Sides.*

Call AB, $a$; CD, $b$; and AD, $x$; and let the given ratio of AC to BC be expounded by that of $m$ to $n$. Hence $BD=a-x$; $\overline{AC}^2$ $(=\overline{CD}^2+\overline{AD}^2)$ $=bb+xx$ and $\overline{BC}^2$ $(=\overline{CD}^2+\overline{BD}^2)$ $=bb+aa-2ax+xx$: and therefore $mm : nn :: bb+xx : bb+aa-2ax+xx$. From which, by multiplying extremes and means,




we have $m^2b^2+m^2a^2-2m^2ax+m^2x^2=n^2b^2+n^2x^2$: whence $\overline{mm-nn}\times xx-2mmax=\overline{nn-mm}\times bb-m^2a^2$; and $xx-\frac{2mma}{mm-nn}\times x=-bb-\frac{m^2a^2}{mm-nn}$: which, solved, gives $x=\frac{mma}{mm-nn}\pm\sqrt{\left|\frac{mna}{mm-nn}\right|^2-b^2}$.

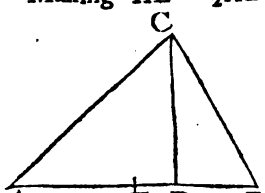
*Geometrically.*

The geometrical construction of this problem is given *by Elem.* 15. 4. *1st edit.* For, if the base AB be divided at E in the given ratio of AC to BC, and, in AB produced, there be taken, EO, a fourth proportional

portional to AE−BE, AE, and BE, *it is there demonſtrated*, that two lines drawn from A and B to meet any where in the circumference of a circle deſcribed through E, from the center O, will be in the ſame given ratio of AE to BE. Whence it is evident that the interſection of the ſaid circumference with a right-line FG, drawn parallel to AB, at the given diſtance DC, will determine the vertex of the triangle.

## PROBLEM XX.

*Having the Baſe* AB, *the Perpendicular* CD, *and the Difference of the two Sides*, AC *and* BC *of a Triangle* ABC; *to find the Sides.*




Making $AE = \frac{1}{2}AB = a$, $CD = b$, $AC - BC = d$, and $ED = x$, we have $AD = a + x$, $BD = a - x$, $AC = \sqrt{b^2 + \overline{a+x}^2}$, and $BC = \sqrt{b^2 + \overline{a-x}^2}$; and conſequently $\sqrt{bb + \overline{a+x}^2} - d = \sqrt{bb + \overline{a-x}^2}$. Which equation, ſquared, gives $bb + \overline{a+x}^2 - 2d\sqrt{bb + \overline{a+x}^2} + dd = bb + \overline{a-x}^2$; and this, by reduction, becomes $4ax + dd = 2d\sqrt{bb + \overline{a+x}^2}$. And this, again ſquared, produces $16a^2x^2 + 8addx + d^4 = 4dd \times \overline{bb + aa + 2ax + xx}$; or, $16a^2x^2 + d^4 = 4dd \times \overline{aa + bb} + 4ddxx$. Whence $x = \sqrt{\frac{4dd \times \overline{aa+bb} - d^4}{16aa - 4dd}}$.

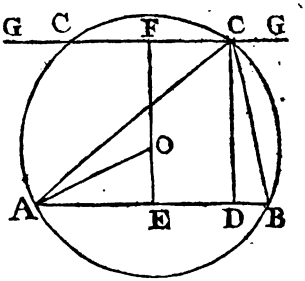
The geometrical conſtruction of this problem being only a particular caſe of a more general one, given at large hereafter (*problem* 49) I ſhall not inſert it here: but obſerve, with reſpect to the algebraical ſolution, that, if $d$ be ſuppoſed to denote the ſum, inſtead of the difference, of the ſides, the value of $x$ (or DE) will be

be given by the very equation above exhibited; as is manifeſt from the proceſs.

## PROBLEM XXI.

*Having the Baſe* AB, *the Perpendicular* CD, *and the Rectangle of the two Sides* AC *and* BC, *of a Triangle* ABC; *to determine the Triangle.*

By retaining the notation of the preceding problem, and making the given rectangle $= c^2$, we have $\sqrt{bb+\overline{a+x}|^2} \times \sqrt{bb+\overline{a-x}|^2} = c^2$; and therefore $\overline{bb+\overline{a+x}|^2} \times \overline{bb+\overline{a-x}|^2} = c^4$; that is $b^4+b^2\times 2a^2+2x^2 + a^4 - 2a^2x^2 + x^4 = c^4$. Whence $x^4+2\overline{bb-2aa} \times x^2 = c^4 - b^4 - 2a^2b^2 - a^4$.




Which, ſolved, gives $x = \sqrt{aa-bb \pm \sqrt{c^4-4a^2b^2}}$.

*Geometrically.*

The magnitude of the rectangle under the two unknown lines, AC and BC, being given, two other lines muſt, therefore, be aſſigned, containing an equal rectangle; whereof one being given, the other will alſo become known. (*Vid. obſervtion* 5, *p.* 88.) But it is known that the rectangle under the, given, perpendicular CD and the diameter of a circle circumſcribing the triangle, is equal to the rectangle under the ſaid, unknown, ſides of the triangle (*Elem.* 25. 3): hence the diameter of the circumſcribing circle is given; and from thence the following conſtruction.

Find a third-proportional to CD and the ſide of the ſquare, expreſſing the magnitude of the propoſed rectangle; and with the half thereof, from the point A (or B) deſcribe an arch, cutting EF, perpendicular to AB,

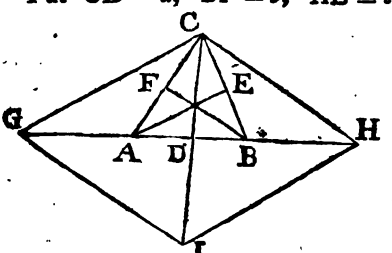
Digitized by Google

AB, in O; from which point, as a center, with the ſame radius, let a circle ACB be deſcribed; then a right-line, GFG, drawn parallel to AB, at the given diſtance DC, will interſect the ſaid circle in the vertex of the triangle.

## PROBLEM XXII.

*The Lengths of three Lines* AE, BF *and* CD, *drawn from the Angles to the Middle of the oppoſite Sides of a Tringle, being given; to find the Sides.*

Put $CD = a$, $BF = b$, $AE = c$, $AB = x$, $AC = y$ and $BC = z$. Since, by a known property of triangles, $\overline{AC}^2 + \overline{BC}^2 = 2\overline{CD}^2 + 2\overline{AD}^2$ (*Elem.* 11. 2) we have $y^2 + z^2 = 2a^2 + \frac{1}{2}x^2$, and therefore $y^2 + z^2 - \frac{1}{2}x^2 = 2a^2$. In the ſame manner




$x^2 + z^2 - \frac{1}{2}y^2 = 2b^2$.

$x^2 + y^2 - \frac{1}{2}z^2 = 2c^2$.

From whence (by taking the former of theſe equations from twice the ſum of the two latter) there comes out $4x^2 + \frac{1}{2}x^2 = 2 \times \overline{2b^2 + 2c^2 - a^2}$: and conſequently $x = \frac{2}{3}\sqrt{2b^2 + 2c^2 - a^2}$. By the ſame argument, $y = \frac{2}{3}\sqrt{2a^2 + 2c^2 - b^2}$; and $z = \frac{2}{3}\sqrt{2a^2 + 2b^2 - c^2}$.

*Geometrically.*

If CG and CH be drawn parallel to AE and BF, meeting AB, produced, in G and H; it is plain, becauſe CE = BE, and CF = AF, that AG = AB = BH; and alſo that CG = 2AE, and CH = 2BF. Therefore, the two ſides CG, CH, and the line CD, biſecting the baſe of the triangle GCH being given, the diagonal CI (= 2CD) of the parallelogram GCHI (as well

Digitized by Google

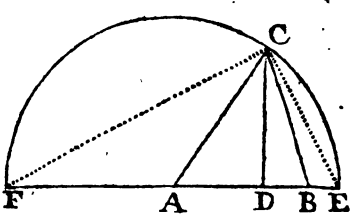
well as the ſides) will be given *(Elem.* 12. 2). Hence, in order to the conſtruction, let a triangle CGI, whoſe three ſides are equal to the doubles of the three given lines, be conſtituted, and draw GDH to biſect CI in D; alſo ſet off DA and DB, each equal to $\frac{1}{3}$ of GD; join A, C, and B, C, *and the thing is done.*

From this conſtruction we have the very ſame numerical ſolution as from the algebraic proceſs: for, ſince, $2\overline{GD}|^2 + 2\overline{CD}|^2 = \overline{CG}|^2 + \overline{CH}|^2$ *(Elem.* 11. 2.) $= 4\overline{AE}|^2 + 4\overline{BF}|^2$, thence is $GD = \sqrt{2\overline{AE}|^2 + 2\overline{BF}|^2 - \overline{CD}|^2}$, and conſequently $AB\ (=\frac{2}{3}GD) = \frac{2}{3}\sqrt{2\overline{AE}|^2 + 2\overline{BF}|^2 - \overline{CD}|^2}$. By the conſtruction it alſo appears that no one of the three given lines muſt be greater than the ſum of the other two.

## PROBLEM XXIII.

*All the Sides of a Triangle* ABC *being given; to find the Perpendicular* CD, *the Segments of the Baſe* AD *and* BD, *together with the Area of the Triangle.*

Put $AC=a$, $AB=b$, $BC=c$, and $AD=x$: then $BD=b-x$; and $c^2-\overline{b-x}|^2\ (=\overline{CD}|^2) = a^2 - x^2$; that is, $c^2 - b^2 + 2bx - x^2 = a^2 - x^2$. Whence $2bx = a^2 + b^2 - c^2$, and $x = \frac{aa+bb-cc}{2b}$.




Now $\overline{CD}|^2 = \overline{AC}|^2 - \overline{AD}|^2 = \overline{AC+AD} \times \overline{AC-AD}$ $= a + \frac{aa+bb-cc}{2b} \times a - \frac{aa+bb-cc}{2b} = \frac{aa+2ab+bb-cc}{2b} \times \frac{-aa+2ab-bb+cc}{2b} = \frac{\overline{b+a}|^2-c^2}{2b} \times \frac{c^2-\overline{b-a}|^2}{2b}$.



Digitized by Google

Hence $CD = \frac{1}{2b}\sqrt{\overline{b+a}^2 - c^2 \times c^2 - \overline{b-a}^2}$; and the area $\left(\frac{CD \times AB}{2}\right) = \frac{1}{4}\sqrt{\overline{b+a}^2 - c^2 \times c^2 - \overline{b-a}^2}$.

*Geometrically.*

From the center A, with the radius AC, let a ſemicircle ECF be deſcribed, cutting AB produced in E and F; ſo that BF may be the ſum, and BE the difference of the ſides AC and AB: alſo let EC and FC be drawn.

Then will $\overline{FC}^2 = FE \times FD$ $(=2AF \times FD)$; and $EC^2 = FE \times ED$ $(=2AE \times ED)$, *by Elem. Corol. to* 20. 4.

Alſo $\overline{BC}^2 = \overline{BF}^2 + \overline{FC}^2$ $(2AF \times FD) - 2FB \times FD = \overline{BF}^2 - 2AB \times FD$ *(Elem. Cor. 2. to* 9. 2): and likewiſe $\overline{BC}^2 = \overline{BE}^2 + \overline{EC}^2$ $(2AE \times ED) - 2EB \times ED = BE^2 + 2AB \times ED$.

Hence it appears that $2AB + FD$ is $= \overline{BF}^2 - \overline{BC}^2$; and $2AB \times ED = \overline{BC}^2 - \overline{BE}^2$: and, conſequently, that $\overline{\overline{BF}^2 - \overline{BC}^2} \times \overline{\overline{BC}^2 - \overline{BE}^2} = 4\overline{AB}^2 \times FD \times ED = 4\overline{AB}^2 \times \overline{DC}^2$ *(by Elem. Cor. to* 19. 4). Therefore $\frac{AB \times DC}{2} = \frac{1}{4}\sqrt{\overline{\overline{BF}^2 - \overline{BC}^2} \times \overline{\overline{BC}^2 - \overline{BE}^2}}$ = the area of the triangle, *as before.*

*From whence it appears, that the area of any triangle will be determined by finding the Differences between the ſquare of any one of its ſides and the Squares of the ſum, and difference, of the other two; and then taking* $\frac{1}{4}$ *of the ſquare root of the product ariſing by the multiplication of the ſaid differences into each other.*

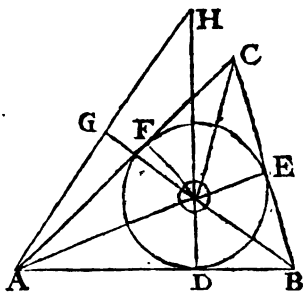
PROBLEM

Digitized by Google

## PROBLEM XXIV.

*Having all the Sides of a Triangle* ABC; *to find the Radius of its inſcribed Circle* DEF.

From the center O, to the angular points and the points of contact, let lines be drawn; and, upon BO produced, let fall a perpendicular AG.




It is plain, in the firſt place (becauſe OD=OE =OF) that AD=AF, BD=BE, and CF=CE: therefore, by addition, BD+CF (=BE+CE) =BC: take each of theſe equal quantities from AB+AC, and there will remain AD+AF= AB+AC−BC; from whence (AD being =AF) we get AD (or AF) = $\frac{AB+AC-BC}{2}$. By ſubtracting of which from AB, and from AC, we alſo have BD = $\frac{AB+BC-AC}{2}$, and CF = $\frac{AC+BC-AB}{2}$.

Moreover, it is evident that the triangles AOG and COF are ſimilar: for the ſum of all the angles at the center, DOE+DOF+FOE being = 4 right angles, the ſum of their halves, BOD+DOA+COF, muſt be = 2 right angles = BOD+DOA+AOG; and conſequently COF=AOG.

Now let the values of AB, BD, and CF (found above) be denoted by $a$, $b$ and $c$, reſpectively; and put OD (OE=OF) =$x$: then, it will be BO ($\sqrt{bb+xx}$): OD ($x$) :: AB ($a+b$) : AG = $\frac{ax+bx}{\sqrt{bb+xx}}$; and BO :



Digitized by Google

BD : : AB : BG $= \frac{ab+bb}{\sqrt{bb+xx}}$. Therefore OG (BG − BO) $= \frac{ab+bb}{\sqrt{bb+xx}} - \sqrt{bb+xx} = \frac{ab-xx}{\sqrt{bb+xx}}$. But, AG : OG : : CF : OF; or $ax+bx : ab-xx : : c : x$; whence $ax^2+bx^2 = abc-cx^2$; and consequently $x = \sqrt{\frac{abc}{a+b+c}}$.

*Geometrically.*

Let DO and AG be produced to meet each other in H. Then, by reason of the similar triangles, it will be, AB : HO (: : AG : OG) : : CF : FO; and therefore by alternation and composition, AB+CF : CF : : HO+FO (HD) : FO (OD) : : HD × OD : $\overline{OD}^2$. But HD × OD is = AD × BD (*Elem.* 24. 3): therefore we have AB+CF : CF : : AD × BD : $\overline{OD}^2 = \frac{AD \times BD \times CF}{AD+BD+CF}$, the very same as before.

From this conclusion, the rule in common practice, for finding the area of a triangle, having the three sides given, is easily deduced: for it is evident that the area of the triangle ABC is equal to the radius (OD) drawn into the half sum of the sides (AD+BD+BF); that is $= \sqrt{AD+BD+CF \times AD \times BD \times CF}$. Where AD, BD, and CF, are the differences between the half sum and each particular side.

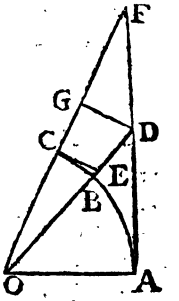
PROBLEM

Digitized by Google

## PROBLEM XXV.

*The Radius of a Circle and the Tangents of two Arcs thereof being given, to determine the Tangent of the Sum of thoſe Arcs.*

Let AB and BC be the propoſed arcs, whereof the given tangents are AD and CE; and let the former of theſe be continued out to meet the radius OC, produced, in F; ſo ſhall AF be the tangent of AC, the ſum of the ſaid arcs.




Now, calling AO, $r$; AD, $m$; CE, $n$; AF, $x$; and FO, $y$; and making DG perpendicular to FO; we have *(by ſimilar triangles)* OF $(y)$ : AO $(r)$ : : DF $(x-m)$ : DG $=\frac{rx-rm}{y}$.

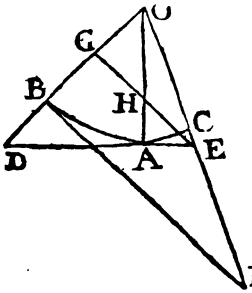
Alſo OF $(y)$ : AF $(x)$ : : DF $(x-m)$ : FG $=\frac{xx-mx}{y}$: from which laſt we have OG $(=$ OF $-$ FG$)$ $=y-\frac{xx-mx}{y}=\frac{yy-xx+mx}{y}=\frac{rr+mx}{y}$ (becauſe $yy-xx=rr$.)

But OG $\left(\frac{rr+mx}{y}\right)$ : DG $\left(\frac{rx-rm}{y}\right)$ : : OC $(r)$ : CE $(n)$, and conſequently $r^2x-r^2m=r^2n+mnx$. Whence $x=\frac{rr\times\overline{m+n}}{rr-mm}$; or AF $=\frac{\overline{AO}^2\times\overline{AD+CE}}{\overline{AO}^2-AD\times CE}$.

*Otherwiſe.*

Digitized by Google

*Otherwiſe.*




Let AD and AE be the tangents of the two arcs AB and AC, and BF that of their ſum BC; alſo let EG be drawn perpendicular to OD, interſecting the radius OA in H. Then, by reaſon of the ſimilar triangles, it will be AO : AD : : AE : AH. And, OH (AO − AH) : DE (: : OG : GE) : : OB (AO) : BF $= \frac{AO \times DE}{AO - AH} = \frac{\overline{AO}^2 \times DE}{AO^2 - AO \times AH} = \frac{\overline{AO}^2 \times \overline{AD + AE}}{AO^2 - AD \times AE}$; becauſe, by the firſt proportion $AO \times AH = AD \times AE$. Which concluſion is the very ſame with that above.

If the tangents of the arcs AC and AB (*fig.* 1.) were to be given, in order to find the tangent of their difference BC; then, by the proportion $\frac{rr + mx}{y}$ : $\frac{rx - rm}{y}$ : : $r$ : $n$ (above derived), we ſhould have $n = \frac{rr \times \overline{x - m}}{rr + mx}$; or $CE = \frac{\overline{AO}^2 \times \overline{AF - AD}}{\overline{AO}^2 + AF \times AD}$.

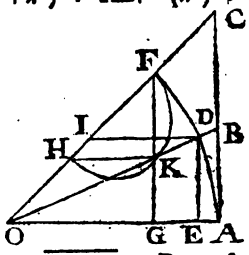
## PROBLEM XXVI.

*The Ratio of the Sines* DE, FG *of two Arcs* AD, AF, *of a given Circle, together with that of their Tangents* AB, AC *being given; to find both the Sines and the Tangents.*

Put the radius $AO = a$; and let the given ratio of AB to AC be that of $m$ to $n$; moreover let DE be to FG

Digitized by Google

FG as $p$ to $q$; and call AB, $x$; and AC, $y$: then *(by similar triangles)* $\overline{OB}|^2\ (a^2+x^2) : \overline{AB}|^2\ (x^2) :: \overline{OD}|^2\ (a^2) : \overline{DE}|^2 = \frac{a^2x^2}{aa+xx}$; and $\overline{OC}|^2\ (a^2+y^2) : \overline{AC}|^2\ (y^2) :: \overline{OF}|^2\ (a^2) : \overline{FG}|^2 = \frac{a^2y^2}{aa+yy}$. Therefore, *by the queſtion*, $p^2 : q^2 :: \frac{a^2x^2}{aa+xx} : \frac{a^2y^2}{aa+yy}$, and




consequently $p^2y^2 \times \overline{a^2+x^2} = q^2x^2 \times \overline{a^2+y^2}$. But, by the queſtion, we alſo have, $m : n :: x : y, = \frac{nx}{m}$; which value, ſubſtituted in the preceding equation, gives $\frac{n^2p^2x^2}{mm} \times \overline{a^2+x^2} = q^2x^2 \times a^2 + \frac{nnxx}{mm}$: whence $n^2p^2 \times \overline{a^2+x^2} = q^2 \times \overline{a^2m^2+n^2x^2}$; and $x = \frac{a}{n} \times \sqrt{\frac{nnpp-mmqq}{qq-pp}}$. From which AC, DE and FG are alſo given.

*Geometrically.*

Since the ratio of AB to AC is given, as $m$ to $n$; GK (ſuppoſing K to be the interſection of FG and OB) will be to GF, in the ſame given ratio: and, if (agreeable to the 3d general obſervation) KH be drawn parallel to AO, meeting OF in H, OH will be to OF *ſtill* in the ſame given ratio.

Again, if DI be drawn parallel to AO, meeting OF in I; then OK : OH (:: OD (OF) : OI :: FG : DE) :: $q$ : $p$; whence OK is alſo given; and from thence the following conſtruction.

In any radius OF of the given circle, take OH to OF in the given ratio of $m$ to $n$; and upon HF let a ſemi-circle be deſcribed: take alſo a fourth proportional

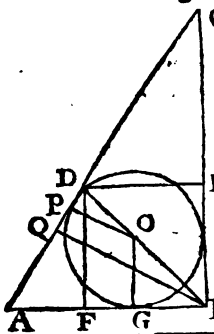
Digitized by Google

portional to $p$, $q$, and OH; with which, as a radius, from the center O, describe an arch, cutting the semi-circle in K; and, having drawn HK, make OA parallel thereto.

## PROBLEM XXVII.

*The Side of the Square, and the Radius of the Circle, inscribed in a right angled Triangle* ABC, *being given; to determine the Triangle.*

Draw the diagonal BD of the square; and from the center O of the given circle, to the points of contact, draw the radii OG and OP; and upon the hypothenuse AC let fall the perpendicular BQ: calling the side of the square, $a$; the radius of the circle, $b$; and AQ, $x$: then because of the parallel lines, we shall have FG $(a-b)$ : BF $(a)$ : : (OD : BD : :) OP $(b)$ : BQ $= \frac{ab}{a-b}$: whence DQ $(=\sqrt{BD^2-BQ^2})$ $=\sqrt{2a^2-\frac{aabb}{\overline{a-b}^2}}$, is also given.




Let it, for brevity-sake, be denoted by $c$; and let BQ $\left(\frac{ab}{a-b}\right)=d$: then we shall have, AQ $(x)$ : BQ $(d)$ : : BQ $(d)$ : CQ $=\frac{dd}{x}$; and also *(by Elem.* 18. 4.) AD $(x+c)$ : CD $\left(\frac{dd}{x}-c\right)$ : : AB : BC : : AQ $(x)$ : BQ $(d)$; whence, by multiplying extremes and means, we get $dx+cd=dd-cx$; and, from thence, $x=\frac{d\times\overline{d-c}}{d+c}$. By means whereof every thing else is readily found.

*Geometrically.*

Digitized by Google

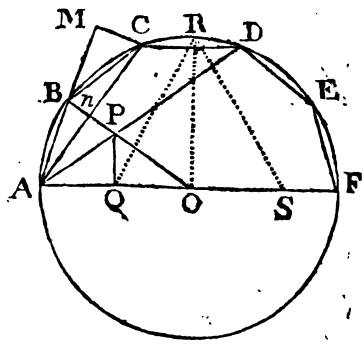
*Geometrically.*

The geometrical conſtruction, and the trigonometrical ſolution of this problem are very eaſy. For, ſince the poſition of the point D with reſpect to the circle, is given; a right-line DPA drawn from thence *(by Elem.* 21. 5.) to touch the circle, will determine the triangle: and then, from the given lines OP and OD, the angle D may be found: by means whereof and ABD (= ½ a right angle) together with the given ſide BD, all the reſt will become known.

## PROBLEM XXVIII.

*To determine the Sides of a regular Pentagon and Decagon, inſcribed in a given Circle.*

Let AB, BC, CD, *&c.* be ſides of the decagon, and AC a ſide of the pentagon: and let AD be drawn, interſecting the radius OB in P. It is evident in the firſt place, that the angles BAP and OAP, ſtanding on the equal arches BD and BF, are equal to one another, and alſo equal, each of them, to the angle AOP, inſiſting on the arch AB *(Elem.* 10. 3.) And ſecondly, that the triangle BAP (as well as APO) is an iſoſceles one, becauſe the perpendicular A*n*C, makes equal angles BAC, DAC with the two ſides AB, AP of the triangle. Hence it appears very plain that all the three lines AB, AP, and OP are equal among themſelves; and likewiſe that AO (OB) : AB (OP) : : OP (AB) : BP *(by Elem.* 18. 4.) ſeeing the angle BAO



is bisected by AP. Moreover, by letting fall the two perpendiculars PQ and CM, upon AO and ABM, the triangles BCM and APQ (as BC is =AB=AP, and the angle MBC=MAD=PAQ) will appear to be equal in all respects; and so, BM being (=AQ) = $\frac{1}{2}$AO, we have $\overline{AC}|^2$ (=$\overline{BC}|^2+\overline{AB}|^2+2BM\times AB$, *Elem.* 11. 2.) =$\overline{BC}|^2+\overline{AB}|^2+AO\times AB$. But, by the above proportion, $\overline{AB}|^2$ is =AO×BP: therefore, by writing AO×BP in the room of $\overline{AB}|^2$, we get $\overline{AC}|^2=\overline{BC}|^2+AO\times AB+AO\times BP=\overline{BC}|^2+\overline{AO}|^2$: whence (BC being first found) AC will also become known.—If AO be now denoted by $a$, and AB by $x$; then from the equality of $\overline{AB}|^2$ and AO×BP, you will have $x^2=a\times\overline{a-x}$; from which $x$ will be found $=\sqrt{\frac{5aa}{4}}-\frac{a}{2}$; and from thence AC (=$\sqrt{xx+aa}$) $=a\sqrt{\frac{5-\sqrt{5}}{2}}$.

As to the geometrical construction, it likewise very easily follows from above. For, since the side of the decagon appears to be equal to the greater part of the radius divided according to extreme and mean proportion; and the square of the side of the pentagon exceeds that of the decagon by the square of the radius; the following solution (given by Dr. *Barrow*, as an improvement upon *Euclid*'s) is manifest.

Draw the radius OR, at right-angles to the diameter AF; and, having bisected the radius AO in Q, let QS be taken, in AF, equal to the distance QR: so shall OF : OS : : OS : FS (*Elem.* 19. 5.) and consequently OS=AB the side of the decagon.

And, because $\overline{RS}|^2$ (supposing RS drawn) is =$\overline{OS}|^2+\overline{OR}|^2$, it is plain also that RS will be equal to the side AC of the pentagon.

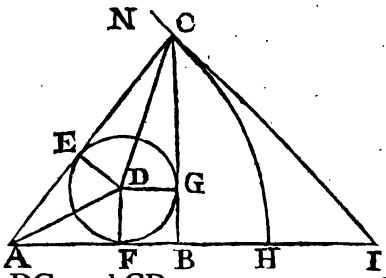
PROBLEM

Digitized by Google

## PROBLEM XXIX.

*Having the Hypothenuſe* AC *of a right-angled Triangle* ABC, *and alſo the Radius of the inſcribed Circle* DEFG; *to find the two Legs* AB *and* BC.

From the center D of the given circle, to the points of contact, let DE, DF, and DG be drawn; alſo draw AD and CD: and put DE (= DG = DF) = $a$, AC = $b$, AB = $x$, and BC = $y$.




It is evident, that CE muſt be = CG = $y - a$; becauſe the right-angled triangles CDE and CDG, having DE = DG, and CD common, are equal in all reſpects. In the very ſame manner is AE = AF = $x - a$.

Therefore $y - a + x - a = b$ (= AC); from which equation we have $x + y = b + 2a$. But, from the property of right-angled triangles, we alſo have $xx + yy = bb$. And, if from the double of this, the ſquare of the former equation be ſubtracted, there will remain $xx - 2xy + yy = bb - 4ab - 4aa$.

From whence, by extracting the ſquare root, on both ſides, we get $x - y = \sqrt{bb - 4ab - 4aa}$. Which laſt equation, added to, and ſubtracted from, the firſt, gives $2x = 2a + b + \sqrt{bb - 4ab - 4aa}$, and $2y = 2a + b - \sqrt{bb - 4ab - 4aa}$.

*Geometrically.*

Seeing the difference between each leg of the triangle and the adjacent ſegment of the hypothenuſe, is equal to the radius of the circle, it is plain that the ſum of the two legs (AB + BC) will exceed the ſum



Digitized by Google

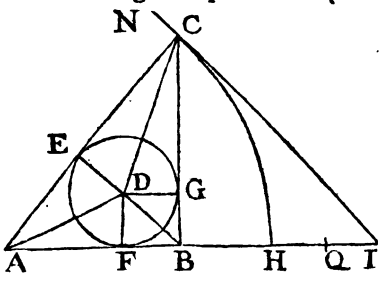
of the two ſegments (or the whole hypothenuſe) by twice the radius; and will therefore be given $=AC+2BF$. Whence, if BI be ſuppoſed equal to BC, AI will likewiſe be given $=AC+2BF$, and the angle BIC = half a right-one. Hence the following conſtruction.

Draw an indefinite line, in which take AH equal to the given hypothenuſe, and HI equal to the diameter of the given circle; alſo make the angle AIN equal to half a right angle; and from the center A, with the interval AH, let an arch be deſcribed, meeting IN in C; from which point upon AP let fall the perpendicular CB; ſo ſhall AB and BC be the two legs required.

## PROBLEM XXX.

*The Perimeter, and the Area, of a right-angled Triangle* ABC *being given; to determine the Triangle.*

Put the given perimeter $(AB+BC+AC)=p$, the area $(\frac{1}{2}AB\times BC)=a^2$; and let half the ſum of the legs AB and BC be denoted by $x$, and half their difference by $y$: then, AB being $=x+y$, BC $=x-y$, and AC $=p-2x$, we ſhall have




$\overline{x+y}\times\overline{x-y}$ $(=AB\times BC)$ $=2a^2$; and $\overline{x+y}^2+\overline{x-y}^2$ $(=AC^2)=\overline{p-2x}^2$; that is, by reduction, $xx-yy=2a^2$; and $2xx+2yy=pp-4px+4xx$.

Now, by the addition of the latter of theſe equations to the double of the former, there ariſes $4xx=pp-4px+4xx+4a^2$: whence $x$ comes out $=\frac{pp+4aa}{4p}=\frac{1}{4}p+\frac{aa}{p}$. From which value that of $y=\sqrt{x^2-2a^2}$, will likewiſe be given.

*Geometrically.*

Digitized by Google

*Geometrically.*

If from the center D, of a circle inſcribed in the triangle, lines be ſuppoſed drawn to the angular points, the propoſed triangle will, by that means, be divided into three others ADB, BDC, and ADC; whoſe baſes are the three ſides of the firſt triangle, and their perpendiculars all radii of the ſaid circle. From which it is evident that the triangle ABC is equal to a rectangle under half the ſum of its three ſides (which we will here expreſs by the given line AQ) and the radius DF of the inſcribed circle; and conſequently that the radius DF will be given by taking a third proportional to AQ and the ſide *(a)* of the ſquare expreſſing the given area. Whence, making QH and QI, each, equal to DF, ſo found; it will appear from the preceding problem that AH will be = the hypothenuſe, and AI = to the ſum of the two legs, of the propoſed triangle: which quantities being both given, the method of conſtruction is manifeſt *from the laſt problem.*

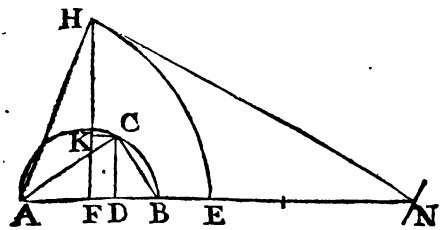
PROBLEM

Digitized by Google

## PROBLEM XXXI.

*The Sum, or Difference of the two Legs* AC *and* BC *of a right-angled Triangle* ABC *being given, together with the Sum, or Difference of the Hypothenuſe* AB *and a Perpendicular* CD *falling thereon from the Right-angle; to find all the Sides of the Triangle.*

Put $AC + BC = s$, $AC - BC = d$, $AB + CD = p$, $AB - CD = q$: then will AC $= \frac{s+d}{2}$, BC $= \frac{s-d}{2}$, AB $= \frac{p+q}{2}$, CD $= \frac{p-q}{2}$: but $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$; and AB × CD (= 2 area ABC) = AC × BC: which, in ſpecies, give $p^2 + 2pq + q^2 = 2s^2 + 2d^2$, and $p^2 - q^2 = s^2 - d^2$.




By adding, and ſubtracting the double of the laſt of theſe equations to and from the former, we have theſe two other equations,

*viz.* $3p^2 + 2pq - q^2 = 4s^2$,
and, $-p^2 + 2pq + 3q^2 = 4d^2$.

From which, when any two of the quantities, $s$, $d$, $p$, $q$, are given, the other two will, eaſily, be determined.

Thus, let $s$ and $p$ be given; then, from the former equation, we have $q^2 - 2pq = 3p^2 - 4s^2$; whence $q^2 - 2pq + p^2 = 4p^2 - 4s^2$, and $p - q = 2\sqrt{p^2 - s^2}$: therefore CD $\left(= \frac{p-q}{2}\right) = \sqrt{p^2 - s^2}$, and $AB = p - \sqrt{p^2 - s^2}$.

If $d$ and $q$ be given; we ſhall have $CD = \sqrt{q^2 - d^2}$; becauſe $p^2 - q^2 = s^2 - d^2$, or $p^2 - s^2 = q^2 - d^2$ (p. above.)

But,

Digitized by Google

But, if $s$ and $q$ be given, then will $3p^2+2pq = 4s^2+q^2$; which, ſolved, gives $p=\frac{2}{3}\sqrt{3ss+qq}-\frac{1}{3}q$.

Laſtly, if $d$ and $p$ be given, we ſhall have $3q^2+2pq = 4d^2+p^2$, and conſequently $q=\frac{2}{3}\sqrt{3dd+pp}-\frac{1}{3}p$.

*Geometrically.*

The very ſame properties whereby the algebraical ſolution is above brought out, lead us alſo to geometrical conſtructions of the ſeveral caſes of the problem under conſideration. But it will be ſufficient, here, to exhibit that of the caſe, wherein the ſum of the legs (AE) and the difference of the hypothenuſe and perpendicular (AF) are ſuppoſed given, being the moſt difficult.

Thus, becauſe $\overline{AC}^2+\overline{BC}^2=\overline{AB}^2$,
and $2AC\times BC$ $(=2AB\times CD)$ $=2AB\times BF$;
we ſhall, by adding theſe equal quantities,
have $\overline{AC}^2+\overline{BC}^2+2AC\times BC$ (or $\overline{AE}^2$, *Elem.* 6. 2.) $=\overline{AB}^2+2AB\times BF=\overline{AB}^2+2AB\times AB-AF=3\overline{AB}^2-2AB\times AF$.

Therefore, if an arch, from the center A, with the radius AE, be deſcribed; and, from its interſection (H) with FH perpendicular to AE, another arch be alſo deſcribed, with the radius $2AE$, cutting AE produced in N; then the hypothenuſe AB of the required triangle, will be $\frac{1}{3}$ of the line AN thus determined.

For $\overline{HN}^2$ $(4\overline{AE}^2)$ being $=\overline{AN}^2+\overline{AH}^2$ $(\overline{AE}^2)$ $-2AN\times AF$ (*Elem.* 10. 2.) and therefore $3\overline{AE}^2=\overline{AN}^2-2AN\times AF$; it is plain, if AB be taken $=\frac{1}{3}AN$, that $3\overline{AE}^2=3AB\times 3AB-6AB\times AF$: and conſequently that $\overline{AE}^2=3\overline{AB}^2-2AB\times AF$, *the very ſame as above.*

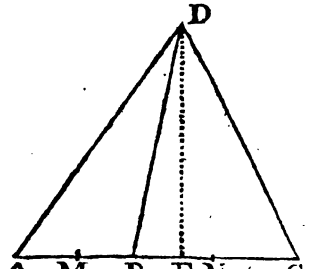
From the value of AB, thus given, what yet remains to be done, will be effected with great facility. For, if in FH there be taken $FK=FB$, and KC be drawn

drawn parallel to AN, interſecting a ſemi-circle deſcribed upon AE, in the point C, that point will, it is evident, be the vertex of the triangle.

## LEMMA.

*If a Line be drawn from the Vertex to any Point in the Baſe of a Triangle, the Sum of the two Solids under the Squares of the two Sides and the alternate Segments of the Baſe, will be equal to the Solid under the whole Baſe and its two Segments, together with the Solid under the ſame Baſe and the Square of the dividing Line.*

Let ACD be any propoſed triangle, and BD the dividing line; then, I ſay, that $\overline{AD}|^2 \times BC + \overline{CD}|^2 \times AB = AC \times AB \times BC + AC \times \overline{BD}|^2$.




For, if AB and CB be biſected in M and N, and a perpendicular DE be let fall upon AC, it is known,

that $\left\{\begin{array}{l}\overline{AD}|^2 - \overline{BD}|^2 = AB \times 2ME \\ \overline{CD}|^2 - \overline{BD}|^2 = BC \times 2NE\end{array}\right\}$ *Elem.* 9. *of* 2. *Cor.* 2.

Whence it follows,

that $\overline{AD}|^2 \times BC - \overline{BD}|^2 \times BC = AB \times BC \times 2ME$;

and $\overline{CD}|^2 \times AB - \overline{BD}|^2 \times AB = AB \times BC \times 2NE$.

Let theſe equal quantities be added together, and the ſums will alſo be equal;

that is, $\overline{AD}|^2 \times BC + \overline{CD}|^2 \times AB - AC \times \overline{BD}|^2$, $(= AB \times BC \times 2MN) = AB \times BC \times AC$; and conſequently $\overline{AD}|^2 \times BC + \overline{CD}|^2 \times AB = AC \times AB \times BC + AC \times \overline{BD}|^2$. *Q. E. D.*

COROL. 1. Hence, if AB=BC, then will $\overline{AD}|^2 + \overline{CD}|^2 = 2AB^2 + 2\overline{BD}|^2$.

COROL.

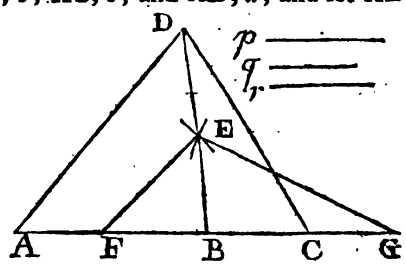
COROL. 2. But, if AD=DC, then we shall have $\overline{AD}|^2 \times BC + \overline{CD}|^2 \times AB = \overline{AD}|^2 \times BC + \overline{AD}|^2 \times AB = \overline{AD}|^2 \times AC$: whence $\overline{AD}|^2 \times AC = AB \times BC \times AC + \overline{BD}|^2 \times AC$; and consequently $\overline{AD}|^2 = AB \times BC + \overline{BD}|^2$.

COROL. 3. Lastly, if the angle ADB = the angle CDB (or AD : CD : : AB : BC, *Elem.* 18. 4.) then $AD \times BC$ being $= CD \times AB$, it follows that $\overline{AD}|^2 \times BC = AD \times CD \times AB$; and that $\overline{CD}|^2 \times AB = AD \times CD \times BC$. Hence $\overline{AD}|^2 \times BC + \overline{CD}|^2 \times AB = AD \times CD \times \overline{AB+BC} = AD \times CD \times AC = AB \times BC \times AC + \overline{BD}|^2 \times AC$ *(p. above)*; and consequently $AD \times CD = AB \times BC + \overline{BD}|^2$.

## PROBLEM XXXII.

*From three given Points*, A, B, C, *in the same Right-line, to draw as many Lines, to meet in a fourth Point* D, *so as to obtain a given Ratio among themselves.*

Call AB, $a$; BC, $b$; AC, $c$; and AD, $x$; and let AD, BD, and CD be, in proportion to one another, as $p$, $q$, and $r$, respectively. Then, BD being $= \frac{qx}{p}$, and CD $= \frac{rx}{p}$, we




shall, *by the preceding lemma*, have $x^2 \times b + \frac{r^2x^2}{pp} \times a = abc + c \times \frac{q^2x^2}{pp}$.

Whence $\overline{bp^2 + ar^2 - cq^2} \times x^2 = abcp^2$;

and consequently $x=p\sqrt{\frac{abc}{bpp+arr-cqq}}$.

After the same manner the problem may be resolved, when, instead of the ratio, the sums, the differences, or the rectangles of the three lines, are given.

*Geometrically.*

If from any point E, in BD, two lines EF and EG be supposed drawn, so as to form the angles BEF and BEG, respectively, equal to BAD and BCD, it is evident, from the similarity of the triangles BEF, BDA, and BEG, BDC,

that $\begin{cases} BE : BF :: BA : BD \\ BE : BG :: BC : BD. \end{cases}$

And, consequently, that BF : BG :: BC : BA. Therefore, if BF be taken =BC, BG will be =AB.

Moreover, from the abovementioned, similar, triangles, we have $\begin{cases} BD : AD \ (:: q : p) :: BF \ (BC) : FE \\ BD : CD \ (:: q : r) :: BG \ (AB) : GE. \end{cases}$

Whence FE and GE are given; and from thence the following construction.

Take BF=BC and BG=AB; also take a fourth proportional to $q$, $p$, and BC, and another to $q$, $r$, and AB; and, with these as radii, from the centers F and G, let two arcs be described; and, to their intersection E, draw BE and FE; also draw AD, making the angle BAD=BEF, so shall its intersection with BE, produced, be the point of concourse required.

The trigonometrical calculation, from this construction, is very short and easy: for, all the sides of the triangle FGE being given, the angle F may be found; then, in the triangle BFE, two sides and the included angle being known, every thing else is readily determined.——It may be observed that there is another construction of this problem; by means of the intersection of two circles, so described that lines drawn from the given points to meet in the peripheries thereof, may obtain the given ratios *(see Elem. 15. 4. 1st edit.)* but the method given above I look upon as preferable.—

As

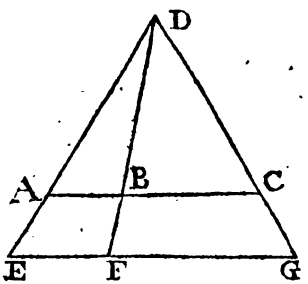
Digitized by Google

As to the limitations, it is plain the problem becomes impoſſible when the two circles, deſcribed from C and F, do not meet each other; that is, when $q$ is given either, leſs than the difference, or greater than the ſum of $\frac{pb}{c}$ and $\frac{ra}{c}$.

## PROBLEM XXXIII.

*The two Sides* AD, CD, *of a Triangle being given in Length, together with the Length of a Line* DB *dividing the Baſe* AC *in a given Ratio; to determine the Baſe, or Line ſo divided.*

Call AD, $a$; BD, $b$; CD, $c$; and AB, $x$; and let the given ratio of AB to BC be that of $m$ to $n$. Hence $BC = \frac{nx}{m}$, and $AC \left(= x + \frac{nx}{m}\right) = \frac{\overline{m+n} \times x}{m}$. And therefore, *by the lemma*, $a^2 \times \frac{nx}{m} + c^2 \times x = \frac{\overline{m+n} \times x}{m} \times \frac{nx}{m} \times x + \frac{\overline{m+n} \times x}{m} \times b^2$.




Whence, by reduction, $mna^2 + m^2c^2 - m \times \overline{m+n} \times b^2 = \overline{m+n} \times nx^2$.

And $x = \sqrt{\frac{mnaa + mmcc}{\overline{m+n} \times n} - \frac{mbb}{n}}$.

If DB be ſuppoſed to biſect the baſe; then, $m$ and $n$ being equal, $x$ becomes $= \sqrt{\frac{aa+cc}{2} - bb}$.

But, if DB be ſuppoſed to biſect the vertical angle, we ſhall have $m : n$ (:: AD : CD) :: $a : c$ (*Elem.* 18. 4.)



Digitized by Google

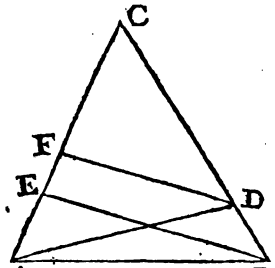
Whence, by writing $a$ and $c$ instead of $m$ and $n$, our equation becomes $x\left(=\sqrt{\frac{a^3c+a^2c^2}{a+c\times c}-\frac{ab^2}{c}}\right)=\sqrt{a^2-\frac{ab^2}{c}}$.

*Geometrically.*

In any right-line EG, at pleasure, let there be taken EF and FG in the given ratio of AB to BC; and, from the points E, F and G, let three lines be drawn to meet in a point D *(by the last problem)* so as to obtain the given ratio of AD, BD, and CD, respectively: then, if DA be taken of the given length, and ABC be drawn parallel to EFG, it is manifest that ADC will be the triangle required.

## PROBLEM XXXIV.

*Supposing two Sides* AC, BC, *of a Triangle* ABC, *to be given in Length; and that two Lines* BE, AD, *drawn from the opposite Angles, to cut off given Segments* AE, BD, *are equal to each other; it is proposed to determine the other Side* AB *of the Triangle.*




Put $CE=a$, $AE=b$, $CD=c$, $BD=d$, $CA=f$, $CB=g$, $AB=x$, and AD (or BE) $=z$. Then, *by the lemma*, we have the two following equations.

$g^2\times b+x^2\times a=fab+f\times z^2$,
$f^2\times d+x^2\times c=gcd+g\times z^2$.

From the former of which, multiplied by $g$, let the latter, multiplied by $f$, be subtracted, and there will arise $bg^3-df^3+agx^2-cfx^2=abfg-cdfg$.

Therefore $x=\sqrt{\frac{ab-cd\times fg+df^3-bg^3}{ag-cf}}$.

*Geometrically.*

Digitized by Google

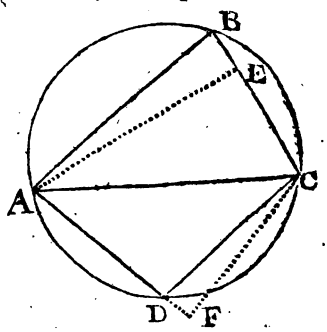
*Geometrically.*

If DF be drawn parallel to BE, the ratio thereof to BE (or AD) will be given, as CD to CB; and CF will likewise be to CE in the same given ratio, and therefore will be given in length. Hence it is evident that the position of the point D with respect to the side AC (first laid down) will be determined by the intersection of two circles; one described from the center C, with the given interval CD; and the other, *by Elem.* 15. 4. *1st edit.* so that lines drawn from F and A, to meet any where in the periphery thereof, may obtain the said given ratio of CD to CB. Through which point, so determined, the other side CB of the triangle must be drawn; and then, AB being joined, the thing is done.

## PROBLEM XXXV.

*All the Sides of a Trapezium* ABCD, *about which a Circle may be described, being given in length; to determine the Diameter of the Circle.*

Let the diagonal AC be drawn, and upon BC and AD let fall the perpendiculars AE and CF. And put $AB=a$, $BC=b$, $CD=c$, $AD=d$, and $BE=x$.




Now the external angle CDF being equal to the internal, opposite, angle B (*Elem.* 17. 3.) the triangles ABE and CDF are similar: and therefore AB ($a$) : BE ($x$) : : CD ($c$) : DF $=\frac{cx}{a}$. But $\overline{AB}^2+\overline{BC}^2-2BC\times BE$ (=

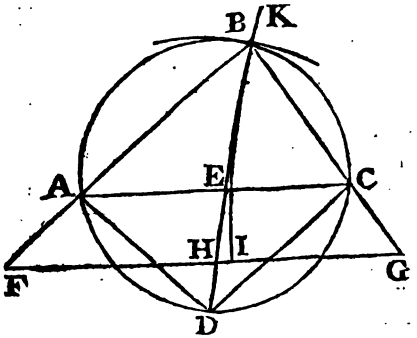
$(=\overline{AC}^2) = \overline{AD}^2 + \overline{CD}^2 + 2AD \times DF$; that is, in ſpecies, $aa+bb-2bx=cc+dd+\frac{2cdx}{a}$.

From which equation $x$ is found $=\frac{aa+bb-cc-dd}{2b+\frac{2cd}{a}}$; whence AE $(=\sqrt{aa-xx})$ and AC $(=\sqrt{aa+bb-2bx})$ will alſo be given: and then it will be AE : AC :: AB : the diameter ſought (*Elem.* 24. 3).

*Geometrically.*

If the two-diagonals of the trapezium be drawn, interſecting each other in E, the triangles ABE and CED, as well as CEB and AED, will be equiangular (*Elem.* 12. 3.)

Whence $\begin{cases} AB : CD :: BE : CE \\ BC : AD :: BE : AE. \end{cases}$




And, ſince the ratios of CE and AE to BE are thus given, it follows, that, if any line FG be drawn parallel to AC, the parts thereof GH and FH, intercepted by BC, BE, and BA (produced) will be to BH in the ſame given ratios,

or that $\begin{cases} AB : CD :: BH : GH \\ BC : AD :: BH : FH; \end{cases}$

and conſequently, that, if BH be taken $=AB$, GH will be given $=CD$, and $FH=\frac{AD \times AB}{BC}$. Whence the following conſtruction.

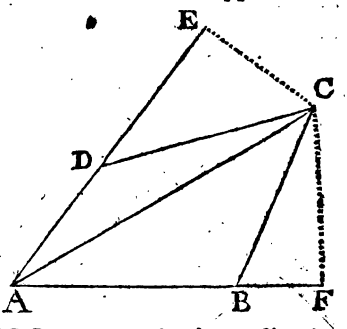
Make

Make FH a fourth-proportional to BC, AB, and AD *(by Elem.* 13. 5); and, in the same line produced, take HG=CD: then *(by Elem.* 15. 4. *1st edit.)* let a circle IK be described, so that two lines drawn from F and G, to meet any where in the periphery thereof, may obtain the given ratio of AB to CB. And from H, as a center, with the radius AB, let another arch be described, intersecting the former in B; draw BF and BG, in which set off BA and BC of the given lengths, then, through the three points A, B, and C, let a circle be described, and the thing is done.

## PROBLEM XXXVI.

*All the Sides, and the Area of a Trapezium* ABCD *being given; to determine the Trapezium.*

Suppose the diagonal AC to be drawn; suppose also CE and CF to be perpendicular to AD and AB: and make AD $=a$, DC $=b$, BC $=c$, AB $=d$, the given area ABCD $=r^2$, DE $=x$, and BF $=y$.




Then will $aq+bb+2ax$ $(=\overline{AC}^2)$ $=cc+dd+2dy$ *(Elem. Cor.* 2. 9. 2.) and $a\sqrt{bb-xx}+d\sqrt{cc-yy}$ $(=2ADC+2ABC)$ $=2r^2$ *(by the question.)*

Put $c^2+d^2-a^2-b^2=2f$; and, by the first equation, you will have $ax-dy=f$.

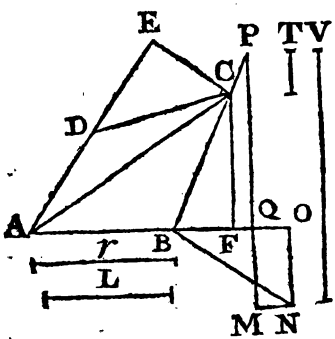
Moreover, by squaring the two last equations, and then adding them together, you will have $a^2b^2+c^2d^2+2ad\sqrt{bb-xx}\times\sqrt{cc-yy}-2adxy=4r^4+f^2$. Which, by dividing by $2ad$, and making $g=\frac{4r^4+f^2}{2ad}-\frac{ab^2}{2d}-\frac{c^2d}{2a}$, is reduced to $\sqrt{bb-xx}\times\sqrt{cc-yy}=g+xy$.

This, squared, gives $b^2c^2-b^2y^2-c^2x^2=g^2+2gxy$: which

which, by writing $\frac{dy+f}{a}$ in the room of, its equal, $x$, will become $b^2c^2 - b^2y^2 - \frac{c^2 \times \overline{dy+f}^2}{aa} = g^2 + 2gy \times \frac{dy+f}{a}$: whence, putting $b^2c^2 - \frac{c^2f^2}{a^2} - g^2 = h$, $\frac{2c^2df}{a^2} + \frac{2fg}{a} = k$, and $b^2 + \frac{c^2d^2}{a^2} + \frac{2dg}{a} = l$, we get $h = ky + ly^2$; and therefore $y = \sqrt{\frac{h}{l} + \frac{kk}{4ll}} - \frac{k}{2l}$.

*Geometrically.*

Becauſe $\overline{AD}^2 + \overline{DC}^2 + 2AD \times DE\ (= \overline{AC}^2) = \overline{AB}^2 + \overline{BC}^2 + 2AB \times BF$ *(Elem. Cor. 2. 9. 2.)* it is plain that $2AD \times DE - 2AB \times BF$ is given $= \overline{AB}^2 + \overline{BC}^2 - \overline{AD}^2 - \overline{CD}^2$.




Find *(by Elem. 2. and 3. 6.)* a line $L$ whoſe ſquare ſhall be $= \overline{AB}^2 + \overline{BC}^2 - \overline{AD}^2 - \overline{CD}^2$; ſo ſhall $2AD \times DE - 2AB \times BF = L^2$, and conſequently $DE - \frac{AB \times BF}{AD} = \frac{L^2}{2AD}$.

But $\frac{AB \times BF}{AD}$, (ſuppoſing PQ perpendicular to AB, and BP a fourth proportional to AD, AB, and BC) appears to be $= BQ$: and $AB \times CF$ (ſince $AD : AB :: BC : BP :: CF : PQ$) will alſo be equal to $AD \times PQ$ *(Elem. 10. 4.)*

Hence

Hence we have $DE-BQ=\frac{L^2}{2AD}$, and $AD\times EC+AB\times CF\ (=2r^2)=AD\times EC+AD\times PQ$: whence, by taking a third proportional *(T)* to 2AD, and *L*, and another *(V)* to $\frac{1}{2}$AD and *r*, there results $DE-BQ=T$; and $EC+PQ=V$.

Therefore the problem is reduced to this; *to find two angles* CDE, PBQ, *ſo that the ſum of their ſines* CE, PQ, *and the Difference of their co-ſines* DE, BQ *(anſwering to given, but unequal, radii* DC, BP*) may be both given quantities.*

And the conſtruction thereof, (which is exceeding evident) will be as follows.

Draw two lines cutting each other at right-angles in M, in which take $MP=V$, the given ſum of the ſines; and $MN=T$, the given difference of the co-ſines: then, from the centers P and N, with the given radii PB $\left(=\frac{AB\times BC}{AD}\right)$ and DC, let two arcs be deſcribed, interſecting each other in B; through which point draw QA parallel to MN; and join B, P, and B, N; ſo ſhall PBQ and NBQ (=CDE) be the two angles required. Which being known, the trapezium itſelf is very eaſily conſtructed.

This problem, it may be obſerved, becomes impoſſible when the two arcs, deſcribed from the centers P and N, do not meet, but fall ſhort of each other; that is, when the given area is greater than that of a trapezium of the ſame given ſides, inſcribed in a circle, determined by the preceding problem.

But, beſides the above, there is another limit, for the leaſt value of the area (except in one particular caſe) and the problem will be impoſſible when one of the two circles falls wholly within the other, as well as when it falls wholly without it: but this laſt depends upon the particular order of joining the given lines; whereas the caſe is otherwiſe with reſpect to the firſt, or greateſt limit.

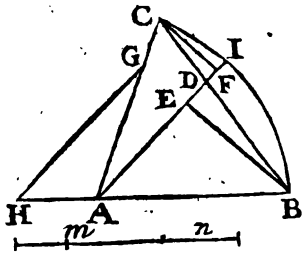


Digitized by Google

## PROBLEM XXXVII.

*To divide a given Angle* BAC *into two Parts* BAI *and* CAI, *ſo that their Sines* BE *and* CF *may obtain a given ratio; ſuppoſe that of* m *to* n.

Put the chord BC of the given angle $=a$, and the part thereof BD, intercepted by AI, $=x$: then, becauſe of the ſimilar triangles BDE and CDF, it will be, BD $(x)$ : CD $(a-x)$ : : BE : CF : : $m$ : $n$, *by hypotheſis.*




Therefore $nx=m\times\overline{a-x}$; and conſequently $x=\frac{ma}{m+n}$.

From whence, and the given angle ABD, the angle BAD will be found.

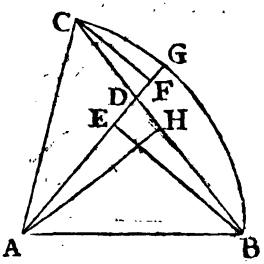
The geometrical conſtruction (like the algebraical operation) may be performed by dividing the ſubtenſe BC in the given ratio of $m$ to $n$: but the following method is preferable.

In AC take AG $=m$; and, in BA produced, take AH $=n$; then a line AE drawn parallel to that joining the points G and H, will divide the angle as required. For, *by trigonometry*, AG $(m)$ : AH $(n)$ : : ſine AHG (BAE) : ſine AGH (CAE): from this conſtruction the numerical ſolution is exceeding eaſy; both the ſides AG and AH, and the included angle HAG being given.

PROBLEM

## PROBLEM XXXVIII.

*To divide a given Angle* BAC *into two ſuch Parts* BAG, CAG, *that the Rectangle under their Sines,* BE *and* CF *(to a given Radius) may be of a given Magnitude.*




Let AH be perpendicular to the chord BC; alſo let BH ( = CH ) = $a$, AH = $b$, BE × CF = $c^2$, and HD = $x$; ſuppoſing AG to interſect BC in D.

Becauſe of the ſimilar triangles DAH, DBE, and DCF,

we have $\begin{cases} \text{AD } (\sqrt{bb+xx}) : \text{AH } (b) :: \text{BD } (a+x) : \text{BE} \\ \text{AD } (\sqrt{bb+xx}) : \text{AH } (b) :: \text{CD } (a-x) : \text{CF.} \end{cases}$

Whence, by compounding the two proportions,

$$bb+xx : bb :: \overline{a+x} \times \overline{a-x} : \text{BE} \times \text{CF} = \frac{bb \times \overline{aa-xx}}{bb+xx} = c^2.$$

From which equation $x$ is found $= b\sqrt{\frac{aa-cc}{bb+cc}}$.

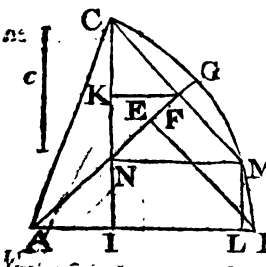
By means of which and the foregoing proportions both BE and CF will become known.



Digitized by Google

*Geometrically.*

If CI be ſuppoſed perpendicular to AB, and FK to CI; then, the oppoſite angles CNF and ANI being equal, their complements NCF and NAI will likewiſe be equal: and therefore, the triangles CFK and ABE being equiangular, we have AB : BE : : CF : FK, or AB × FK = BE × CF = $c^2$ *(according to obſervation 5.)* whence FK is given. Therefore if, from the extremity of the chord CFM, there be drawn MN perpendicular to CI, the length thereof, being twice that of FK, will alſo be given; and, from thence the following conſtruction.




Take IL a third proportional to ½AB and the ſide (*c*) of the given ſquare, expreſſing the magnitude of the propoſed rectangle: draw LM perpendicular to AB, meeting the arch BC in M; then a line AG drawn to biſect MC will divide the angle BAC as required.

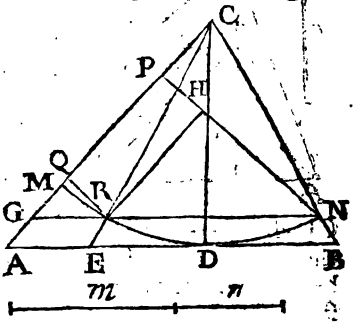
The numerical ſolution, from hence, is very conciſe and eaſy: for, having found the value of IL (by dividing the meaſure of the given rectangle by half the radius) let the co-ſine AI of the whole, given, angle be added thereto; then the ſum AL will be the co-ſine of (BM) the difference of the two, required, parts.—This problem becomes impoſſible when IL is given greater than IB; that is, when the rectangle propoſed is greater than half the rectangle under AB and BI.

PROBLEM

## PROBLEM XXXIX.

*To divide a given Angle* MCN *into two ſuch Parts* MCD, NCD, *that their Tangents* AD, BD, *may obtain a given Ratio; ſuppoſe that of* m *to* n.

Put the radius CD $= a$, the tangent of the given angle MCN $= b$, and that (AD) of the part MCD $= x$. Then it will appear, from Problem 25, that the tangent BD of the remaining part NCD is truly expreſſed by $\frac{a^2 \times \overline{b-x}}{aa+bx}$.




Hence we have $m : n :: x : \frac{a^2 \times \overline{b-x}}{aa+bx}$: and therefore $mn \times \overline{aa+bx} = ma^2 \times \overline{b-x}$. From which $x$ is found

$$= a\sqrt{\frac{m}{n} + \frac{\overline{m+n} \times a^2}{2nb}\Big|^2} - \frac{\overline{n+m} \times a^2}{2nb}.$$

*Geometrically.*

The ratio of AD to BD being given, as $m$ to $n$, the ratio of their ſum AB to their difference AE (ſuppoſing DE $=$ DB) will be given, as $m+n$ to $m-n$. And, if NG be drawn parallel to BA, meeting CE and CA in R and G, the whole line NG will be to the part GR in the ſame ratio of $m+n$ to $m-n$: and it is evident, that, if two perpendiculars NP and RQ be let fall from the points N and R upon AC, they will likewiſe be in that ratio. Whence the following conſtruction.

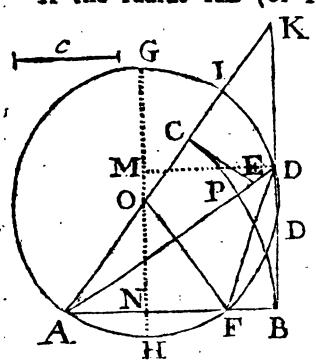
In NP, perpendicular to MC, take PH a fourth proportional to $m+n$, $m-n$, and PN; draw HR parallel

rallel to CM interfecting the arch MN in R, and draw CD to biſect NR, *and the thing is done.*

For the trigonometrical calculation, it will be $m+n : m-n :: NP : RQ$; that is, as $m+n$ is to $m-n$, ſo is the ſine of the whole, given, angle to the ſine of the difference of its two, required, parts; whence the parts themſelves will be known.

## PROBLEM XL.

*To divide a given Angle* BAC *into two Parts* BAP *and* CAP, *ſo that the Rectangle under their Tangents* BD *and* CE *(to a given Radius* AB*) may be of a given Magnitude.*




If the radius AB (or AC) be denoted by $a$; the tangent BK of the given angle BAC, by $b$; and the tangent BD of the part BAP, *by* $x$; then the tangent CE of the remaining part CAP will be repreſented by $\frac{a^2 \times b-x}{aa+bx}$ *(ſee the note to Prob.* 25.) And we ſhall, therefore, have $\frac{a^2 \times b-x \times x}{aa+bx}$ $(=BD \times CE) = c^2$. Whence $bx - xx = \frac{cc \times aa+bx}{aa} = cc + \frac{bc^2x}{aa}$: which, by making $d = b - \frac{bcc}{aa}$, becomes $dx - xx = cc$; whence $x = \frac{1}{2}d \pm \sqrt{\frac{1}{4}dd - cc}$.

*Geometrically.*

Digitized by Google

*Geometrically.*

If a line DF be drawn to make the angle BDF equal to CAE, the triangles BDF and CAE will be ſimilar; and we ſhall, therefore, have AC×BF=BD×CE=$c^2$ *(according to Obſervation* 5. *p.* 88) whence BF, and conſequently AF, will be given. Moreover the ſum of the angles BAD and BDF being equal to the given angle BAC *(by hypotheſis)* and the ſame ſum +ADF equal to a right-angle *(by Elem. Cor.* 4. *to* 9. 1.) it is evident that ADF is equal to the difference between the ſaid given angle BAC and a right-one.

Therefore, having taken BF a third proportional to AB and the ſide *(c)* of the given ſquare, make the angle AFO = the angle FAC; and from (O) the interſection of AC and FO, let a circle be deſcribed through A and F, interſecting the tangent BK in D, and D; from either of which points draw AD, *and the thing is done.*

In order to the trigonometrical calculation, let the diameter GH be drawn to biſect the arch AF in H, and let DM be perpendicular thereto: then, having found AF, it will be, as AN ($\frac{1}{2}$AF) : DM (NB) :: ſine AH (co-ſine NAO) : ſine GD (=co-ſine $\frac{1}{2}$DD) =co-ſine of the difference of the two required angles BAD and DAI; whence, as their ſum is given, the angles themſelves will be known.

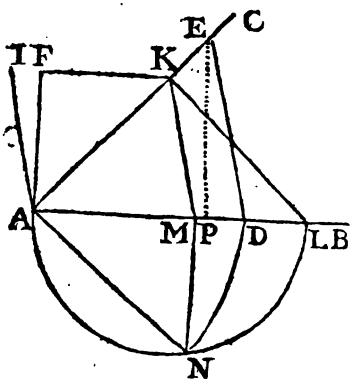
This problem becomes impoſſible, when the circle OAFG neither cuts, nor touches, the line BK; that is, when the given rectangle is greater than the ſquare of the tangent of half the propoſed angle, and the angle itſelf is acute; or, when the ſaid rectangle is leſs than the ſquares of half the tangent of the propoſed angle, and the angle itſelf is obtuſe.

PROBLEM

Digitized by Google

## PROBLEM XLI.

*To draw a Line* DE *parallel to a given Line* AI, *so as to interſect two other Lines* AB, AC, *given by Poſition, and thereby form a Triangle* ADE *of a given Magnitude.*




Let the given area of the triangle be denoted by $a^2$; and let the ſines of the given angles DAE, ADE, and AED (to the radius $r$) be expreſſed by $m$, $n$, and $p$, reſpectively: then, making EP perpendicular to AD, and calling AD, $x$; we have *(by plane trigonometry)* as $p : m :: x : DE = \frac{mx}{p}$; and $r : n :: DE : EP = \frac{mnx}{rp}$.

Hence $\frac{mnx}{rp} \times \frac{x}{2} = a^2$:

and therefore $x = \sqrt{\frac{2rpaa}{mn}} = a\sqrt{\frac{2rp}{mn}}$.

*Geometrically.*

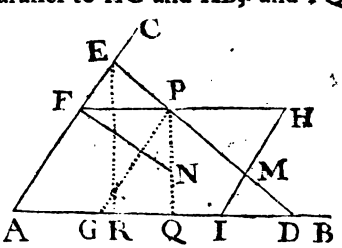
Let AF, perpendicular to AB, be the ſide of a ſquare equal to the triangle ADE; then, if AL be taken = 2AF, and FK be drawn parallel to AB, *&c.* it is evident that the triangle ALK, being $= \overline{AF}^2$, will alſo be equal to the triangle ADE. Moreover, by making KM parallel to AI, it will be $\overline{AD}^2 : \overline{AM}^2 :: ADE\ (ALK) : AMK$ *(Elem.* 24. 4.) :: AL : AM *(Elem.* 7. 4.) :: AL × AM : $\overline{AM}^2$. And conſequently $\overline{AD}^2 = AL \times AM$

× AM. Therefore, AD being a mean proportional between AL and AM, upon the former of these let a semi-circle ANL be described, intersecting MN, perpendicular to AB, in N; make AD = the chord AN, and draw DE parallel to AI, and the thing is done (*Elem.* 19. 4.)

## PROBLEM XLII.

*Through a given Point* P, *between two Lines* AB *and* AC, *given by Position, to draw another Line* DE, *so that the Triangle* ADE, *formed thereby, may be of a given Magnitude.*

Let PG and PF be parallel to AC and AB, and PQ and ER perpendicular to AB: also let AG and PQ (which are given by the position of P) be denoted by $a$ and $b$, respectively. Then, calling AD, $x$, and denoting the given area by $c^2$, we shall have, DG $(x-a)$ : PQ $(b)$ :: AD $(x)$ : ER $= \frac{bx}{x-a}$:




and therefore $\frac{bx}{x-a} \times \frac{x}{2}$ $(= \text{ER} \times \frac{1}{2}\text{AD}) = c^2$.

Whence $xx - \frac{2c^2x}{b} = -\frac{2ac^2}{b}$; and consequently $x = \frac{c^2 + c\sqrt{c^2 - 2ab}}{b}$.

*Geometrically.*

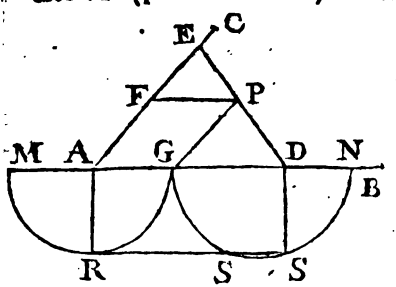
If upon AF, a parallelogram AFHI be constituted, to contain the given area, it will appear, by taking away AFPMI from each of the equal quantities AH and ADE, that the remainders PHM and PFE + IDM will likewise be equal: and so, these three triangles

being all similar to one another, it follows that $\overline{PH}^2$ will be $=\overline{PF}^2+\overline{ID}^2$ (*Elem.* 24. 4.) Whence this construction.

From the center F, with the interval PH, let an arch be described, cutting PQ perpendicular to AB, in N; make ID=PN, and draw DPE, and the thing is done. For it is evident that $\overline{PH}^2$ ($\overline{FN}^2$) is $=\overline{PF}^2+\overline{DI}^2$ ($\overline{PN}^2$.)—Hence it also appears that the problem will be impossible when PH is less than PF; or when the triangle proposed to be constructed is less than twice the parallelogram AFPG.

## PROBLEM XLIII.

*Through a given Point* P, *between two Lines* AB, AC *given by Position, to draw another Line* DE, *so that the Sum of the Parts,* AD+AE, *cut off thereby, from the two former, may be a given Quantity.*




Let PF (parallel to AB) be denoted by $a$, and PG (parallel to AC) by $b$; also let AD + AE = $c$, and AD = $x$: then, by reason of the parallel lines, we shall have DG $(x-a)$ : PG $(b)$ : : AD $(x)$ : AE $=\frac{bx}{x-a}$.

And therefore $\frac{bx}{x-a}+x$ $(=AE+AD)$ $=c$.

From which, by reduction, $xx-\overline{c+a-b}\times x=-ac$;

and consequently $x=\frac{c+a-b}{2}\pm\sqrt{\frac{\overline{c+a-b}^2}{4}-ac}$.

*Geometrically.*

Digitized by Google

*Geometrically.*

The triangles DGP and PFE being ſimilar, the rectangle under DG and FE will therefore be equal to the given rectangle under AF and AG = $\overline{AR}^2$; by taking AM = AF, and making AR a mean proportional between AM and AG. Moreover, if MN be taken = the given ſum of AD and AE, it is manifeſt that the ſum of DG and FE (whoſe rectangle is above given) will, alſo, be given = GN.

Therefore, having deſcribed a ſemi-circle upon GN, let RS be drawn parallel to AB, interſecting the periphery thereof in S; from which point, upon AB, let fall the perpendicular SD, and through P draw DPE, and the thing is done. For DG × DN = $\overline{DS}^2$ = $\overline{AR}^2$ = AM × AG = AF × AG = DG × EF: whence EF = DN; and conſequently AD + AE (= AD + AF + DN) = MN.

It is plain, this problem becomes impoſſible, when RS falls below the ſemi-circle GN; or, when the ſum of AD and AE is ſuppoſed to exceed that of AG and AF by leſs than the double of a mean proportional between AG and AF.

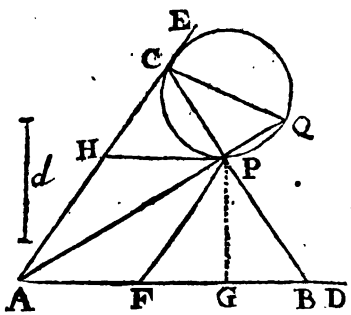
Much after the ſame manner the problem will be reſolved, when the difference, the ratio, or the rectangle of AD and AE is given.



Digitized by Google

## PROBLEM XLIV.

*Through a given Point,* P, *betwixt two Right-lines* AD, AE, *given by Position, so to draw a Right-line* BPC *that the Rectangle* (BP × CP) *under the Parts thereof, intercepted by that Point and those Lines, may be of a given Magnitude.*




Let PF and PH be parallel to AC and AB, and PG perpendicular to AB: then, calling PF, $a$; PH, $b$; FG, $c$; and BF, $x$; we have $\overline{BP}^2 = aa + xx - 2cx$ (*Elem.* 10. 2.)

And, by similar triangles, BF ($x$) : BP :: PH ($b$) : CP $= \frac{b}{x} \times$ BP. Consequently BP × CP $= \frac{b}{x} \times \overline{BP}^2 = \frac{b}{x} \times \overline{aa + xx - 2cx}$. Hence, if the given value of BP × CP be denoted by $d^2$, then will $\frac{d^2 x}{b} = aa + xx - 2cx$: which equation, solved, gives $x = \frac{dd}{2b} + c \pm \sqrt{\overline{\frac{dd}{2b} + c}^2 - a^2}$.

*Geometrically.*

The rectangle under two unknown lines being given, another line must therefore be found, or assumed, under which and some given line in the figure, an equal rectangle may be contained. (*Vid. Obser.* 5. *p.* 88.)

As,

Digitized by Google

As, in the preſent caſe, the line AP is given, both in length and poſition, let the rectangle under it, and a part, PQ, of the ſame line produced, be therefore aſſumed $=BP\times CP$; then the conſequence will be, *that*, beſides obtaining $PQ=\frac{d^2}{AP}$, the triangles PQC and PBA (ſuppoſing QC drawn) will alſo be ſimilar becauſe $AP : BP :: CP : PQ$.) And ſo, the angle PCQ being $=$ the given angle PAB, it is evident that a ſegment of a circle deſcribed upon PQ *(by Elem.* 22. 5.) capable of containing the ſaid given angle, will interſect AE in the point (or points) required.

This problem will, it is manifeſt, be impoſſible, when the circle, deſcribed as above, falls ſhort of AE; or, according to the algebraic ſolution, when $\left|\frac{dd}{2b}+c\right|^2-a^2$ is negative; that is, when the propoſed rectangle is leſs than $2b\times a-c$, or, its equal $2PH\times PF-FG$.

PROBLEM

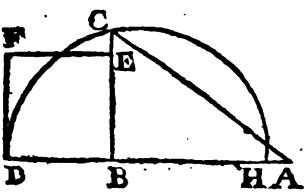
Digitized by Google

## PROBLEM XLV.

*The Area* ($a^2$) *of a right-angled Triangle* ABC, *whose Sides are in Arithmetical Progreſſion, being given, to determine the Triangle.*

Put the greater leg AB$=x$, and the common difference $=y$; ſo ſhall BC$=x-y$, and AC$=x+y$; and therefore

$$\left\{\begin{array}{l}\overline{x+y}^2=\overline{x-y}^2+x^2\\ \frac{1}{2}x\times\overline{x-y}=a^2\end{array}\right\}$$ by the queſtion.




From the former of which equations we have $xx+2xy+yy=2xx-2xy+yy$; and conſequently $4y=x$: whence, by ſubſtituting for $x$ in the ſecond equation, we get $2y\times 3y=a^2$: from which $y$ is given $=\sqrt{\frac{a^2}{6}}$; and $x\ (=4y)=4\sqrt{\frac{a^2}{6}}$. Therefore BC$=3\sqrt{\frac{a^2}{6}}$, AB$=4\sqrt{\frac{a^2}{6}}$, and AC$=5\sqrt{\frac{a^2}{6}}$.

*Geometrically.*

It is well known that $\overline{AC+BC}\times\overline{AC-BC}$ $(=\overline{AC}^2-\overline{BC}^2)$ is $=\overline{AB}^2$ *(Elem. Corol. to* 8. 2.) And, by the queſtion, AC + BC is $=2$AB (becauſe AB is an arithmetical mean between AC and BC.) Therefore $2AB\times\overline{AC-BC}=\overline{AB}^2$; and conſequently AC $-$ BC $=\frac{1}{2}$AB: take theſe equal quantities from the equal quantities AC + BC and 2AB, and the remainders, 2BC and $1\frac{1}{2}$AB, will be equal; and conſequently $\frac{3}{4}AB=4BC$. But $\frac{1}{2}AB\times BC\ (=\frac{2}{3}BC\times BC)=\overline{BD}^2\ (=a^2)$; and therefore $\overline{BC}^2=\frac{1}{6}\overline{BD}^2$.

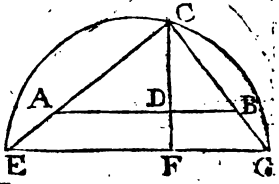
Hence

Hence, if BH be taken to BD ($a$) in the proportion of 3 to 2, a mean proportional BC between DB and HB, will be the leſſer leg of the triangle required; whoſe greater leg BA, being in proportion thereto, as 4 to 3, is alſo given from hence.

## PROBLEM XLVI.

*The Area* ($a^2$) *of a right-angled Triangle* ABC, *whoſe Sides are in Geometrical Proportion, being given; to determine the Triangle.*

Make $AC = x$; then $BC = \frac{2aa}{x}$, and $AB = \sqrt{x^2 + \frac{4a^4}{xx}}$.




Therefore, $\frac{2a^2}{x} : x :: x : \sqrt{x^2 + \frac{4a^4}{xx}}$, *by the queſt.*

Hence $x^2 = \frac{2a^2}{x}\sqrt{x^2 + \frac{4a^4}{xx}}$; $x^4 = \frac{4a^4}{xx} \times \frac{x^4 + 4a^4}{xx}$; $x^8 - 4a^4x^4 = 16a^8$; $x^4 - 2a^4 = a^4\sqrt{20}$; and $x = a \times \sqrt[4]{2 + \sqrt{20}}$.

*Geometrically.*

Since, *by hypotheſis*, AB : AC :: AC : BC, therefore is $\overline{AB}^2 : \overline{AC}^2 :: \overline{AC}^2 : \overline{BC}^2$ (*Elem. Cor.* 1. *to* 11. 4.)

But $\overline{AC}^2$, ſuppoſing CD perpendicular to AB, is equal to AB × AD; and $\overline{BC}^2$ equal to AB × BD (*Elem. Cor. to* 19. 4.)

Therefore $\overline{AB}^2 : AB \times AD :: AB \times AD : AB \times BD$; or, AB : AD :: AD : BD.

Whence the following conſtruction.

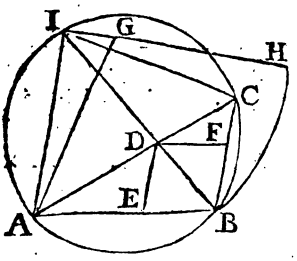
Draw any line EG, at pleaſure; which divide at F (*by Elem.* 19. 5.) according to extreme and mean proportion (ſo that EG : EF :: EF : GF); erect the perpendicular

perpendicular FC, and upon EG let a semi-circle be described intersecting it in C; join E, C, and G, C; and let AB *(by Prob.* 41.) be drawn, parallel to EG, to cut off the given area ABC ($=a^2$), *and the thing is done.* For it is manifest that AB is divided, by CDF, in the same proportion with EG; or that, AB : AD :: AD : BD, *as above.*

## PROBLEM XLVII.

*Supposing one Side* AC, *and the opposite Angle* ABC *of a Triangle to be given, together with the Side* DE, *or* DF, *of the inscribed Rhombus* EF; *to find from thence the other two Sides of the Triangle.*

Conceive a circle ABCI to be described about the triangle, and the diagonal of the rhombus produced to meet the circumference thereof in I: also let AI and CI be drawn; which will be equal to each other *(Elem.* 12. 3.) as being the subtenses of the equal angles IBA and IBC, formed by the diagonal and sides of the rhombus: and since the arcs AI and CI, as well as the chords, are equal, the angles IAD and IBA, insisting upon them, must likewise be equal; and consequently the triangle IAD similar to the triangle IBA.




But the vertical angle AIC of the isosceles triangle ACI (as well as the base AC) is given, being =2 right-angles − ABC *(Elem. Cor.* 17. 3.) whence IA will be given *(by plane trigonometry.)*

Put, therefore, IA=$a$, BD=$b$, and IB=$x$; then, from the similarity of the triangle above specified, we shall have, $x-b$ (ID) : $a$ (IA) :: $a$ (IA) : $x$ (IB) whence $xx-bx=aa$; and consequently $x=\sqrt{aa+\frac{1}{4}bb}+\frac{1}{2}b$. From which, and the known values of IA and the angle ABI, the value of AB, *&c.* will also be known. *Geometrically.*

Digitized by Google

*Geometrically.*

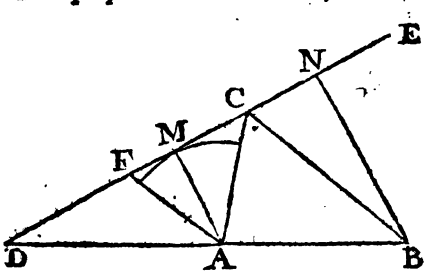
The geometrical conſtruction hereof, as the rect angle BI × DI (by the foregoing proportion) is given = AI², is obvious from Problem 6; and is thus,

Having, upon the given ſide AC, deſcribed a ſegment of a circle ABC capable of containing the given angle ABC (*Elem.* 22. 5); and, in the other ſegment, AIC, conſtituted the iſoſceles triangle AIC, make IG perpendicular to IA, and equal to half (BD) the given diagonal of the rhombus; alſo, in IG produced, take GH equal to the diſtance GA; and, through H, from the center I, let an arch be deſcribed interſecting the circle ABC in B; then draw BA and BC, and the thing is done.

## PROBLEM XLVIII.

*From two given Points* A *and* B, *to draw two Lines* AC *and* BC, *to meet in a Right-line* DE *given by Poſition, and form an Angle* ACB *of a given Magnitude.*

Make AM and BN perpendicular to DE; and let AF be ſuppoſed parallel to BC: then, calling AM, $a$; BN, $b$; MN, $c$; and MC, $x$; it will be as $b : c - x$ (NC) :: $a$ : MF $= \frac{ac - ax}{b}$




But MF and MC are tangents of the angles MAF and MAC; which angles together (becauſe of the parallel lines AF and BC) make an angle CAF equal to the given angle ACB: therefore, if the tangent of the ſaid given angle, to the radius AM, be de-



Digitized by Google

noted by $t$, it will appear, *from Problem* 25, that $\frac{\overline{MC+MF}\times AM^2}{AM^2-MC\times MF}=t$.

This, in ſpecies, gives $\frac{\overline{x+\frac{ac-ax}{b}}\times a^2}{a^2-\frac{acx-axx}{b}}=\frac{\overline{bx-ax+ac}\times a}{ab-cx+xx}=t$.

Which, by reduction, becomes $\frac{\overline{b-a}\times ax}{t}+\frac{aac}{t}=xx-cx+ab$. Whence, making $c+\frac{\overline{b-a}\times a}{t}=d$, and $a\times\overline{\frac{ac}{t}-b}=f$, we have $f=xx-dx$; and conſequently $x=\frac{1}{2}d\pm\sqrt{f+\frac{1}{4}dd}$.

When the line MN is parallel to that joining the given points A, B; then, BN *(b)* becoming =AM *(a)* we have, in this caſe, $d=c$, and $f=\frac{aac}{t}-a^2$; and therefore $x=\frac{c}{2}\pm\sqrt{\frac{aac}{t}-aa+\frac{1}{4}cc}$. Which may ſerve as a theorem for finding the ſegments of the baſe of a triangle (and conſequently the triangle itſelf) when the whole baſe, the perpendicular, and the vertical angle are given.

As to the geometrical conſtruction of the general problem, it is extremely obvious; ſince a ſegment of a circle deſcribed upon AB *(by Elem.* 22. 5.) capable of containing the given angle, will interſect DE in the point (or points) required. Whence it alſo appears that the problem will be impoſſible when the circle falls ſhort of the line FE; and, conſequently, that the angle ACB will be the greateſt poſſible when the circle touches the ſaid line; or, when DC is a mean proportional between DA and DB *(Elem. Cor. to* 22. 3.)

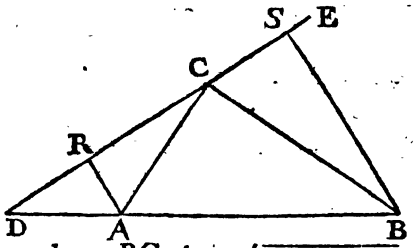
PROBLEM

Digitized by Google

## PROBLEM XLIX.

*To find a Point* C, *in a Right-line* DE, *given by Position; ſo that two Lines* CA, CB *being drawn from thence to two given Points* A *and* B, *the one* BC *ſhall exceed the other* AC *by a given difference* d.

Let AR and BS be perpendicular to DE; and let theſe perpendiculars, together with RS (which are all known from the poſition of DE) be denoted by $a$, $b$, and $c$, reſpectively: then putting $AC = x$, we have $RC \left(= \sqrt{\overline{AC}^2 - \overline{AR}^2}\right) = \sqrt{xx - aa}$; and $SC \left(= \sqrt{\overline{BC}^2 - \overline{BS}^2}\right) = \sqrt{xx + 2xd + dd - bb} = c - \sqrt{xx - aa}$.




From which equation, by ſquaring both ſides thereof, we get $xx + 2dx + dd - bb = cc - 2c\sqrt{xx - aa} + xx - aa$.

This contracted becomes $2c\sqrt{xx - aa} = bb - aa - dd + cc - 2dx$, or, $\sqrt{xx - aa} = m - nx$; by dividing by $2c$, and putting $\frac{bb - aa - dd + cc}{2c} = m$, and $\frac{d}{c} = n$.

Therefore, by ſquaring again, $xx - aa = mm - 2mnx + nnxx$; whence $1 - nn \times xx + 2mnx = aa + mm$, or $xx + \frac{2mnx}{1 - nn} = \frac{aa + mm}{1 - nn}$.

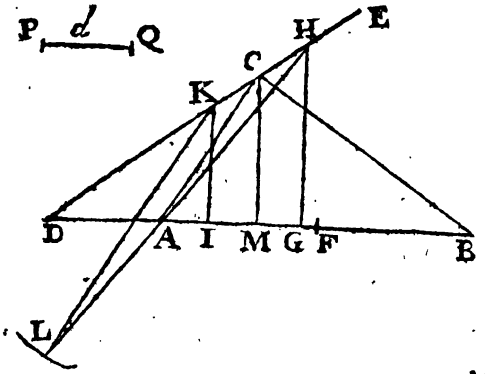
And conſequently $x = \sqrt{\frac{aa + mm}{1 - nn} + \frac{m^2 n^2}{\overline{1 - nn}^2}} - \frac{mn}{1 - nn}$.



Digitized by Google

*Geometrically.*

If AB be drawn, and biſected in F, and CM be ſuppoſed perpendicular thereto, it is well known that $FM = \frac{\overline{BC+AC} \times \overline{BC-AC}}{2AB}$ (*Elem. Cor.* 2. 9. 2.) Therefore, ſince $BC-AC=PQ$, and, conſequently, $BC+AC = PQ+2AC$, it follows that $FM = \frac{\overline{PQ+2AC} \times PQ}{2AB} = \frac{\overline{PQ}^2}{2AB} + \frac{AC \times PQ}{AB}$.




But, ſince both AB and PQ are given, the value of $\frac{\overline{PQ}^2}{2AB}$, (being a third proportional to 2AB and PQ) is alſo given. Let this, therefore, be expreſſed by FG, ſo that FM may be $= FG + \frac{AC \times PQ}{AB}$: then it is plain that $GM = \frac{AC \times PQ}{AB}$; or that, GM is to AC in the given ratio of PQ to AB.

But the ratio of GM to HC (ſuppoſing GH perpendicular to AB) is alſo given by the poſition of the line DE; whence the ratio of HC to AC (where both belong to the ſame triangle AHC) will be exhibited: for, taking GI = PQ, and making IK perpendicular to AB, it will be $\frac{HC}{HK} = \frac{GM}{GI} \left(= \frac{GM}{PQ}\right) = \frac{AC}{AB}$, *p. above)* and conſequently HC : AC : : HK : AB.

But,

But, if KL be ſuppoſed parallel to AC, meeting HA produced (if need be) in L, it is manifeſt that HC : AC : : HK : KL. Therefore KL = AB, whence the following conſtruction.

Having taken $FG=\frac{\overline{PQ}^2}{2AB}$, GI = PQ, and erected the perpendiculars GH and IK, and alſo drawn HA *(as above ſpecified)*, from the center K, with the interval AB, let an arch be deſcribed, cutting HA produced in L; and, having drawn LK, make AC parallel thereto; which will cut the given line DE in the point required.

The trigonometrical calculation, from this conſtruction, may be as follows.

Having computed FG $\left(=\frac{\overline{PQ}^2}{2AB}\right)$ and ſubtracted it from FD and FA, the remainders GD and GA will be given; and it will then be as GD : GA : : *tang.* DHG (or *co-tang.* D) : *tang.* AHG; whence the angle LHK is likewiſe known. And, ſince *ſin.* DHG (or *co-ſin.* D) : *rad.* : : IG (PQ) : HK, the value of HK (as well as thoſe of KL and the angle KHL) will be known: from which the other two angles of the triangle HKL being found, the angle DAC (= ECA − D = HKL − D) will alſo be obtained.

After the very ſame manner the problem may be reſolved, when the ſum, inſtead of the difference, of the lines AC and BC is given.

As to the reſtrictions of the laſt problem, it is evident that the given difference muſt never exceed the diſtance AB: but when the line DE paſſeth between the points A and B, the limit will be ſtill leſs; but is eaſily determined, in any caſe, from the given poſition of DE. It is a little remarkable, that the above ſolution fails in that particular caſe, only, wherein the general problem becomes moſt ſimple; that is, when DE is perpendicular to AB. But here the operation will, alſo, become more ſimple and expeditious: for the

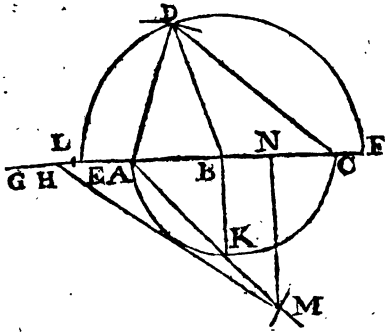
Digitized by Google

the position of MC (with which DE is supposed to coincide) being actually given, the length of $AC = \frac{GM \times AB}{PQ}$ *(p. above)* will also be known. And, if with this, as a radius, an arch be described, from the center A, it will intersect the perpendicular MC (or DC) in the point required.

## PROBLEM L.

*From two given Points* A, C, *in the Diameter* EF *of a given Semi-circle, to draw two Lines to meet in the Circumference, so that one of them* CD *may exceed the other* AD *by a given Difference, not greater than the Distance of the two given Points.*

The radius BD being supposed drawn; put $AB = a$, $BC = b$, $AC = c$, $BD = r$, $AD = x$, and $CD = x + d$; $d$ being the given difference.




Then, *by the Lemma at p.* 128, we shall have $x^2 \times b + \overline{x+d}|^2 \times a = abc + cr^2$; or, $bx^2 + ax^2 + 2adx + ad^2 = abc + cr^2$.

Whence $x^2 + \frac{2adx}{c} = ab + r^2 - \frac{add}{c}$,

and consequently $x = \sqrt{ab + r^2 - \frac{abdd}{cc}} - \frac{ad}{c}$.

*Geometrically.*

The geometrical construction is also deducible from the *Lemma* above specified. For it is evident from thence, that $\overline{AD}|^2 \times \frac{BC}{AB} + \overline{CD}|^2$ is $= AC \times BC +$

AC

$\frac{AC \times \overline{BD}^2}{AB} = AC \times \overline{BC} + AC \times BG$ (by taking $BG = \frac{\overline{BD}^2}{AB}$) $= AC \times \overline{BC + BG} = AC \times CG$.

Therefore, having taken BG a third proportional to AB and BD; and CL a mean-proportional between AC and CG; draw BK perpendicular to GF, meeting the circumference of a semi-circle, described upon AC, in K; and, having drawn AKM, and taken AH equal to the given difference of AD and CD, upon H as a center, with the radius LC, let an arch be described, intersecting AK in M; from which point upon GF let fall a perpendicular MN; then, if from A, with the radius AN, another arch be described, it will intersect the arch of the given semi-circle in the point, D, required.

For $AB : BC :: \overline{AB}^2 : AB \times BC$ $(= \overline{BK}^2$, *Elem. Cor.* 19. 4. $:: \overline{AN}^2$ $(\overline{AD}^2) : \overline{NM}^2 = \overline{AD}^2 \times \frac{BC}{AB}$.

And $\overline{NM}^2$ $(\overline{AD}^2 \times \frac{BC}{AB})$ $+ \overline{HN}^2 = \overline{HM}^2 = AC \times CG$ *(by constr.)* $= \overline{AD}^2 \times \frac{BC}{AB} + \overline{CD}^2$ *(p. above.)*

Therefore $CD = HN = AN + AH = AD + AH$; and consequently $CD - AD = AH =$ the given difference, *by construction.*

The method of calculation, from this construction, is sufficiently easy: for having computed BG $(= \frac{\overline{BD}^2}{AB})$ and HM $(= \sqrt{CA \times \overline{CB + BG}})$ and also the angle BAK (which is had by the proportion AB : BC :: squ. rad. : squ. tang. BAK); you will then have, in the triangle HAM, two sides and one angle; whence every thing else is readily determined.

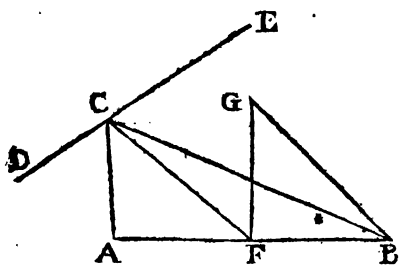
PROBLEM

Digitized by Google

## PROBLEM LI.

*From the Middle* F, *and the two Extremes* A *and* B, *of a given Right-line* AB, *to draw three Lines to meet in a Point* C, *in a Right-line* DE *given by Position, ſo as to be in Geometrical Proportion, or ſo that* AC : FC :: FC : BC.

Becauſe $\overline{BC}^2 + \overline{AC}^2 = 2\overline{FC}^2 + 2\overline{BF}^2$ (*Elem.* 11. 2.) And $BC \times AC = \overline{FC}^2$ (*by Hyp. and Elem.* 10. 4.) it is evident that $\overline{BC}^2 - 2BC \times AC + \overline{AC}^2 = 2\overline{BF}^2$, or $\overline{BC - AC}^2 = 2\overline{BF}^2 = \overline{BG}^2$, by taking BG as the diagonal of the ſquare whoſe ſide is BF.




Hence BC − AC is given = BG; and ſo the caſe in queſtion is reduced to Problem 49; to which I ſhall therefore refer for the remaining part of the ſolution.——It appears from hence that a point in the circumference of a given circle, whoſe center is in the line AB, may be ſo determined, *by the laſt problem*, that three lines drawn from thence to the three given points A, F, B, ſhall be in geometrical proportion.

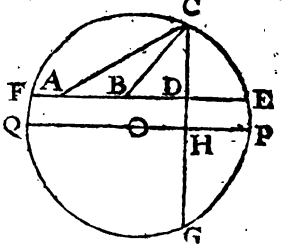
PROBLEM

Digitized by Google

## PROBLEM LII.

*From two given Points* A, B, *within a given Circle, to draw two Lines* AC, BC, *to meet in the Periphery thereof, ſo that the Sum of their Squares may be a given Quantity.*

Through the given points let EF be drawn, meeting the circumference of the circle in E and F; parallel to which, draw the diameter PQ; and let the chord CG be drawn to cut EF and PQ, at right-angles, in D and H.




Put $EF = a$, $EA = b$, $EB = c$, $DH = d$, $ED = x$, and $DC = y$; and let the given quantity, $\overline{AC}^2 + \overline{BC}^2$, be denoted by $e^2$: then will $DF = a - x$, $DA = b - x$, $DB = c - x$, $DG = y + 2d$.

But $ED \times DF = CD \times DG$ (*Elem.* 21. 3.)

And $\overline{DA}^2 + \overline{DC}^2 + \overline{DB}^2 + \overline{DC}^2 = \overline{AC}^2 + \overline{BC}^2$ (*Elem.* 8. 2.) Which, in ſpecies,

give $\begin{cases} x \times \overline{a-x} = y \times \overline{y+2d} \\ \overline{b-x}^2 + y^2 + \overline{c-x}^2 + y^2 = e^2. \end{cases}$

or $\begin{cases} ax - xx = yy + 2dy \\ 2xx + 2yy - 2bx - 2cx = ee - bb - cc. \end{cases}$

Whence, by adding the double of the former equation to the latter, we get $2ax - 2bx - 2cx = ee - bb - cc + 4dy$; and conſequently $y = \frac{bb+cc-ee}{4d} + \frac{a-b-c}{2d} \times x = f + gx$, by making $\frac{bb+cc-ee}{4d} = f$, and $\frac{a-b-c}{2d} = g$.



Digitized by Google

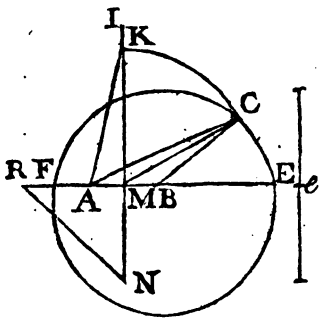
Now, the value of $y$ thus found being ſubſtituted in the firſt equation, there ariſes $ax-xx=ff+2fgx+gg xx+2df+2dgx$;

or, $1+gg \times xx+2dg+2fg-a \times x=-ff-2df$.

From which, by making $m=1+gg$, $n=a-2dg-2fg$, and $p=f\times\overline{f+2d}$, the value of $x$ is found $=\frac{n}{2m}\pm\sqrt{\frac{nn}{4mm}-\frac{p}{m}}$.

*Geometrically.*

That the *locus* of the vertex of a triangle, whereof the baſe and ſum of the ſquares of its two ſides are given, is the circumference of a circle, deſcribed from the middle of the baſe as a center, it is evident, *from Elem.* 11. 2; becauſe the line drawn from the vertex to the middle of the baſe, is an invariable quantity.



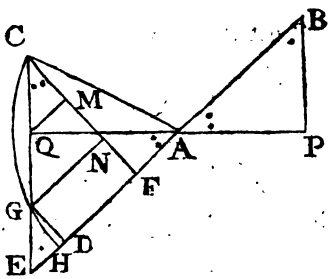

Therefore, having biſected AB with the perpendicular IMN, take MN and MR equal, each, to half the ſide *(e)* of the given ſquare; draw NR, and, from A to the perpendicular MI, draw AK equal to NR; and upon the center M, at the diſtance MK, let an arch be deſcribed; which will meet the circumference of the given circle in the point C, required. For AC, BC, and MC being drawn, it is evident that $\overline{AC}^2+\overline{BC}^2=2\overline{AM}^2+2\overline{CM}^2=2\overline{AM}^2+2\overline{MK}^2=2\overline{AK}^2=2\overline{RN}^2=4\overline{RM}^2=e^2$. In this problem it is requiſite that MC $(=\sqrt{\frac{1}{2}e^2-\overline{AM}^2})$ ſhould be greater than the difference, and leſs than the ſum, of the radius of

of the given circle and the distance of the point M from its center.

## PROBLEM LIII.

*To draw a Line* PQ *through a given Point* A, *ſo that the Rectangle of two Perpendiculars, falling thereon from two other given Points,* B *and* C, *may be of a given Magnitude.*

Let BA and CQ be produced to meet in E; and upon BE let fall the perpendicular CF; which, and the ſegment AF, are both given, by the poſition of the propoſed points.




Put, therefore, CF $=a$, AF $=b$, AB $=c$, and EF $=x$: then will CE $=\sqrt{aa+xx}$: and it will be *(by ſimilar triangles)*,

as CE : EF : : AB : BP $=\frac{cx}{\sqrt{aa+xx}}$;

and CE : EF : : AE : EQ $=\frac{x\times b+x}{\sqrt{aa+xx}}$:

whence CQ (=CE−EQ) $=\sqrt{aa+xx}-\frac{bx+xx}{\sqrt{aa+xx}}=\frac{aa-bx}{\sqrt{aa+xx}}$:

and, conſequently, BP×CQ $=\frac{cx\times\overline{aa-bx}}{aa+xx}=g^2$; ſuppoſing $g^2$ to denote the given magnitude of the propoſed rectangle.

From this equation, by making $d=\frac{aac}{bc+gg}$, and $f=$

$\frac{aag}{bc+gg}$, the value of $x$ is found $=\frac{1}{2}d\pm\sqrt{\frac{1}{4}dd-fg}$. Whence every thing elſe is readily determined.

*Geometrically.*

If QM be ſuppoſed parallel to AB, the triangle CQM will, it is plain, be ſimilar to ABP; and conſequently $AB\times QM=BP\times CQ$ (*Elem.* 24. 3.) $=g^2$: whence QM is given; and, from thence the following conſtruction.

Find a third-proportional to AB and the ſide (*g*) of the given ſquare; and, in BF produced, take FD equal to the double thereof; draw DG parallel to FC, interſecting the circumference of a circle, deſcribed from the center A with the radius AC, in the point G; and, having drawn CG, draw PQ perpendicular thereto.

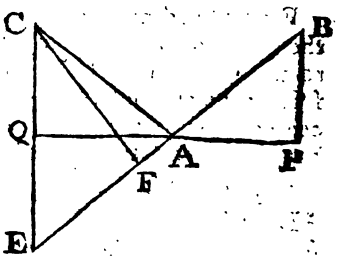
For the trigonometrical calculation, it will be AF : AD (AF+FD) : : co-ſ. HGC : co-ſ. HG, the difference between the angles BAP (HAQ) and CAQ: from which, as their ſum is given, the angles themſelves will be known.—This problem becomes impoſſible when FD is greater than FH; that is, when the propoſed rectangle is greater than half the rectangle under AB and FH.

PROBLEM

Digitized by Google

## PROBLEM LIV.

*Two Lines* AB, AC, *drawn from the ſame Point* A, *being given both in Poſition and Length; to draw another Line* PQ *through that Point, ſo that two Perpendiculars* BP, CQ, *falling thereon from the Extremes of the two given Lines, may form two Triangles* ABP, ACQ, *equal to each other.*




Let BA and CQ be produced to meet in E; and upon BE let fall the perpendicular CF: and put $CF = a$, $AF = b$, $AB = c$, and $EF = x$. The areas of ſimilar triangles being, in proportion, as the ſquares of their homologous ſides; and both the triangles AEQ, ABP, being ſimilar to ECF, we have $\overline{CE}^2$ $(aa + xx)$ : $\frac{1}{2}CF \times EF$ $(\frac{1}{2}ax)$ : : $\overline{AB}^2$ $(c^2)$ : area ABP $= \frac{\frac{1}{2}ac^2x}{aa+xx}$.

And, $aa + xx : \frac{1}{2}ax : : \overline{x+b}^2$ $(= \overline{AB}^2)$ : area AEQ $= \frac{\frac{1}{2}ax \times \overline{x+b}^2}{aa+xx}$. This, taken from the area AEC $(= \frac{1}{2}a \times \overline{x+b})$ leaves the area ACQ $= \frac{\frac{1}{2}a \times \overline{x+b} \times \overline{aa-bx}}{aa+xx}$.

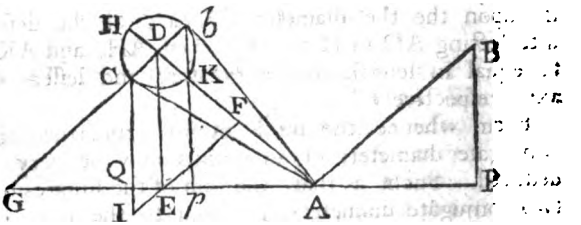
Which being equal to ABP, by hypotheſis, we therefore have $\overline{x+b} \times \overline{aa-bx} = c^2x$: from whence, making $d = \frac{cc-aa}{b} + b$, $x$ is found $= \sqrt{\overline{aa+\frac{1}{4}dd}} - \frac{1}{2}d$.

*Geometrically.*

Digitized by Google

*Geometrically.*

If A*b* be ſuppoſed perpendicular, and equal, to AB, and *bp* perpendicular to PQ, the triangle A*bp*, being ſimilar to ABP, will alſo be equal to it; and conſequently equal to ACQ: and, if theſe equal triangles be, ſucceſſively, taken from the trapezium AQC*b*, the remainders *pb*CQ, and AC*b* will likewiſe be equal.




Hence it appears that the parallelogram under C*b* and DE, whoſe angle is CDE (ſuppoſing DE parallel to, and an arithmetical mean between, CQ and *bp*) is equal to the given triangle AC*b*; and conſequently the altitude thereof equal to half that of the triangle. Whence it is evident that the point E muſt fall, ſomewhere, in a line FI drawn through the middle of AD (or AC) parallel to *b*C (*Elem.* 2. 2.) But the point E, ſince the angle AED is a right-one, will likewiſe fall in the circumference of a ſemi-circle deſcribed upon the diameter AD (*Elem.* 13. 3.) And therefore FE, being a radius, muſt be equal to AF; and conſequently DG=AD; ſuppoſing DC produced to meet AQ in G.

Therefore, in order to the geometrical conſtruction, having made A*b* perpendicular, and equal to, AB, and drawn AD to the middle of C*b* (*as above intimated*) let DG, in DC produced, be taken equal to AD; and from G, through A, draw GP, *and the thing is done.*

It often happens that the demonſtration of a geometrical conſtruction, to be the moſt neat and elegant, proceeds upon principles very different from thoſe whereby we firſt arrived at ſuch conſtruction. The caſe

case above is an instance of it: where, from the similar triangles, it is manifest that GQ (Ap) : QC : : Gp (AQ) : pb, and therefore $\frac{1}{2}AQ \times QC = \frac{1}{2}Ap \times pb = \frac{1}{2}BP \times AP$. *Q. E. D.*

Note. If AB and AC be two semi-conjugate diameters of an ellipsis, then the line PAQ, determined as above, will be the position of the greater axis: and, if, upon the the diameter C*b*, a circle be described intersecting AD in H and K; then AH, and AK will be equal in length to the greater, and lesser, semi-axis, respectively.

From whence the most useful properties of the conjugate diameters of an ellipsis may be very easily deduced. Such, as that, the sum of the squares of any two conjugate diameters, is equal to the sum of the squares of the two axes: and that, any parallelogram described about the conjugate diameters of an ellipsis is equal to the rectangle under the two axes; and so forth. But these are matters not altogether proper to be insisted on in this place.

However it will not be improper to observe, that the last problem is always possible, except in those two cases, wherein the given lines are perpendicular, and form one continued line.

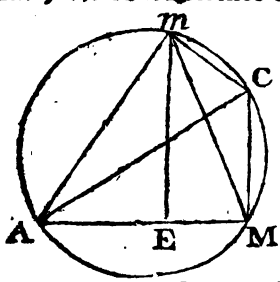


Digitized by Google

## PROBLEM LV.

*If, at the Extremes,* M *and* m, *of two given Right-lines* AM *and* Am, *making a given Angle* MAm, *two Perpendiculars,* MC *and* mC *be erected; it is propoſed to find the Diſtance of their Interſection,* C, *from* A *the given angular Point.*

Since the angles AMC and A*m*C are both right ones, the circumference of a circle, deſcribed upon the diameter AC, will paſs through M and *m*. Therefore, if M*m* be drawn, and a perpendicular *m*E be let fall upon AM, the triangles A*m*C and *m*EM will appear to be equiangular; becauſe the angles AC*m* and EM*m*, inſiſting on the ſame arch A*m*, are equal; and A*m*C is equal to *m*EM, being both right angles.




If now the ratio of *m*E to A*m* (which is given becauſe the angle EA*m* is given) be denoted by that of $s$ to $r$; and the ratio of AE to A*m* by that of $c$ to $r$; we ſhall have $AE = \frac{c}{r} \times Am$; and therefore $\overline{Mm}^2$ $(= \overline{AM}^2 + \overline{Am}^2 - 2AM \times AE) = \overline{AM}^2 + \overline{Am}^2 - \frac{2c}{r} \times AM \times Am$: whence, by reaſon of the ſimilar triangles above ſpecified, it will be, $\overline{mE}^2 : \overline{mA}^2$ $(:: s^2 : r^2)$ $:: \overline{Mm}^2 : \overline{AC}^2 = \frac{r^2}{s^2} \times \overline{AM}^2 + \overline{AM}^2 - \frac{2rc}{ss} \times AM \times Am$; whence AC is given.

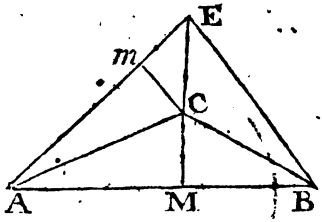
As to the geometrical conſtruction, it is indicated by the conditions of the problem, without any ſort of argumentation.

PROBLEM

## PROBLEM LVI.

*To find a Point* C, *from whence three right-lines drawn to ſo many given Points* A, B, *and* E, *ſhall obtain the Ratio of three given Quantities* a, b, *and* c, *reſpectively.*

The given points being joined; make AB $=f$, AE $=g$, and AC $=x$: then, BC being $=\frac{bx}{a}$, and EC $=\frac{ex}{a}$ *(by hypotheſis)* we alſo have AM $\left(=\frac{1}{2}\text{AB} + \frac{\text{AC}^2-\text{BC}^2}{2\text{AB}}\right) = \frac{f}{2} + \frac{\overline{aa-bb}\times x^2}{2aaf}$ *(Elem.* 9. 2. *Cor.* 2.); and Am $\left(=\frac{1}{2}\text{AE} + \frac{\text{AC}^2-\text{EC}^2}{2\text{AE}}\right) = \frac{g}{2} + \frac{\overline{aa-ee}\times x^2}{2aag}$; ſuppoſing CM and C*m* to be perpendicular to AB and AE, reſpectively.




Hence, putting $\frac{aaf}{aa-bb} = h$, $\frac{aag}{aa-ee} = k$, and $z = \frac{f}{2} + \frac{\overline{aa-bb}\times x^2}{2aaf}$ $(= \text{AM}) = \frac{f}{2} + \frac{xx}{2h}$, we get $xx = 2hz - fh$; and therefore Am $\left(= \frac{g}{2} + \frac{xx}{2k}\right) = \frac{g}{2} + \frac{2hz-fh}{2k} = l + \frac{hz}{k}$; by making $l = \frac{g}{2} - \frac{fh}{2k}$.

Now, ſince *(by the laſt problem)* $\overline{\text{AM}}^2 + \overline{\text{Am}}^2 - \frac{2c}{r}\times \text{AM}\times \text{Am} = \frac{ss}{rr}\times \overline{\text{AC}}^2$ (where $s$ and $c$ denote the ſine and co-ſine of the given angle MA*m*, to the radius $r$) we ſhall, from hence, by ſubſtituting the above

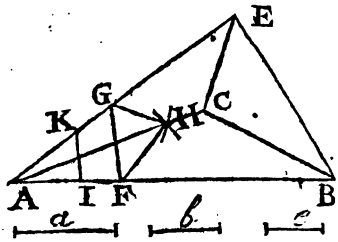
above values of AM, An, and AC, obtain the following equation,

$$zz+l+\frac{bz}{k}\Big|^2-\frac{2cz}{r}\times\overline{l+\frac{bz}{k}}=\frac{ss}{rr}\times\overline{2bz-fh}.$$

From the resolution of which the value of $z$, and from thence the position of the point D, will be determined.

*Geometrically.*

If, in AB, there be taken AF=$a$, and FH be drawn, to make the angle AFH = ACB and meet AC in H; then, the ratio of FH to AF (by reason of the similar triangles AFH, ACB) being the same with that of BC to AC, it is evident that FH is given =$b$.




Moreover, if HG be drawn, making the angle AHG =AEC, it will likewise appear, that both HG and AG are given.

For, since AG : AH : : AC : AE,

And AF : AH : : AC : AB;

it follows that AG : AF : : AB : AE; whence AG is given: and then it will be $a : e$ (: : AC : EC) : : AG : HG. Whence HG is given; and, from thence, the following

*Construction.*

Take AF=$a$, and make the angle AFG=AEB; also take AI=$e$, and draw IK parallel to FG; moreover, from the centers F and G, with the intervals $b$ and AK, describe two arcs, and from the point H of their intersection draw HF and HA, then a line, BC, drawn to make the angle ABC=AHF, will cut AH, produced, in the point required.

As to the trigonometrical solution, it is too obvious, from the construction, to need an explanation. But it

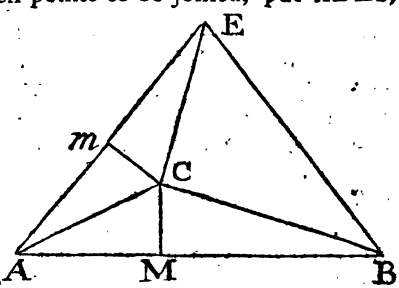
Digitized by Google

it will be proper to take notice that the problem itſelf becomes impoſſible, when the two arcs, deſcribed from the centers F and G, neither cut, nor touch, each other; that is, when the diſtance FG is, either, greater than the ſum, or leſs than the difference, of *b* and AK.

## PROBLEM LVII.

*Three Points*, A, B, *and* E, *being given, to determine a fourth Point* C, *ſo that Lines* (AC, BC, EC) *drawn from thence to the three former, may have given Differences.* (Provided the Difference of no two of the ſaid Lines be given greater than the Diſtance of the two given points from whence they are drawn.)

Suppoſing the given points to be joined, put AB = *a*, AE = *b*, AC = *x*, BC = *x* + *p*, and EC = *x* + *q* (where *p* and *q* repreſent the given differences.) Then, if upon AB and AE, the perpendiculars CM and C*m* be let fall, it will be (by a known property of triangles) AB *(a)* : BC + AC (2*x* + *p*) : : BC − AC *(p)* : BM − AM = $\frac{2px+pp}{a}$ : whence AM = $\frac{1}{2}a - \frac{2px+pp}{2a}$ : and, by the very ſame argument, A*m* = $\frac{1}{2}b - \frac{2qx+qq}{2b}$. Which two values, by putting $\frac{1}{2}a - \frac{pp}{2a} = f$, and $\frac{1}{2}b - \frac{qq}{2b} = g$ (for the ſake of brevity) will become $f - \frac{px}{a}$, and $g - \frac{qx}{b}$.



Moreover, if the ſine, and the co-ſine of the given angle MA*m*, to the radius *r*, be denoted by *s* and *c*, reſpectively, it will appear, from the problem preceding the laſt, that $\overline{AM}^2+\overline{Am}^2-\frac{2c}{r}\times AM\times Am=\frac{ss}{rr}\times\overline{AC}^2$; or, (in ſpecies) $\overline{f-\frac{px}{a}}^2+\overline{g-\frac{qx}{b}}^2-\frac{2c}{r}\times\overline{f-\frac{px}{a}}\times\overline{g-\frac{qx}{b}}=\frac{s^2x^2}{r^2}$. Which equation may be reduced to $\frac{pp}{aa}+\frac{qq}{bb}-\frac{2cpq}{rab}-\frac{ss}{rr}\times x^2-\frac{fp}{a}+\frac{gq}{b}-\frac{cgp}{ra}-\frac{cfq}{rb}\times 2x=-f^2-g^2-\frac{2cfg}{r}$. Whence, by ſubſtituting for the coefficients of the powers of *x*, &c. the value of *x*, will be found, and from thence the poſition of the point C.

*Geometrically.*

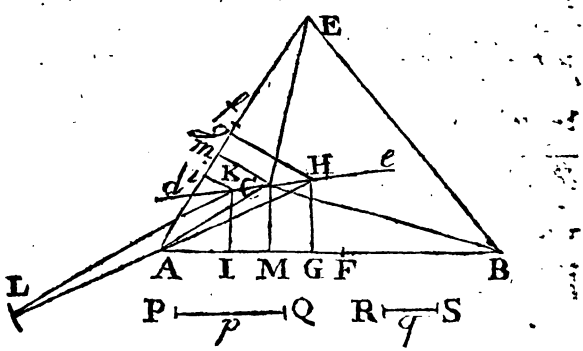
If AB be biſected in F, and FG be taken a third proportional to 2AB and PQ (BC−AC), it is evident, *from Problem* 49, that $GM=\frac{AC\times PQ}{AB}$.

And, for the very ſame reaſons, if AE be biſected in *f*, and *fg* be taken a third proportional to 2AE and RS (EC − AC) we ſhall alſo have $gm=\frac{AC\times RS}{AE}$. Whence it appears that GM is to *gm*, in the given ratio of $\frac{PQ}{AB}$ to $\frac{RS}{AE}$, or of PQ to $\frac{AB\times RS}{AE}$, or, laſtly, of GI to *gi*; by taking GI=PQ, and $gi=\frac{AB\times RS}{AE}$.



Digitized by Google

If therefore GH, and *g*H, and alſo IK and *i*K, be drawn, interſecting in H and K, it is manifeſt that the required point C muſt fall, ſomewhere, in the right-line *de* paſſing through H and K; ſince, in this caſe (and no other) it will be GM : GI (:: HC : HK) :: *gm* : *gi*; or, alternately, GM : *gm* :: GI : *gi*. By




reaſoning in the very ſame manner, from the triangles ACB and BEC, the poſition of another right-line, wherein the point C falls, may alſo be determined; whoſe interſection with *de* will conſequently be the point required.

But, inaſmuch as the caſe, from the poſition of *de* thus given, is now reduced to our 49th *Problem*, the remaining part of the ſolution is likewiſe given, *from thence*, by a different method: according to which, and what is above demonſtrated, we have the following

*Conſtruction.*

Having taken *fg* a third proportional to 2AE and RS; and *gi* a fourth proportional to AE, AB, and RS; and having alſo taken FG a third proportional to 2AB and PQ, and ſet off GI=PQ, and drawn the right-line *de* through the interſections of the perpendiculars

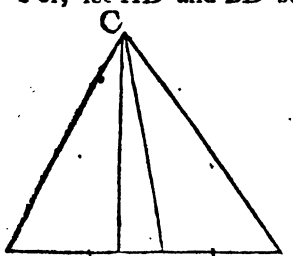
diculars GH, gH; IK, iK, *as above ſpecified*; let an arch of a circle, from the center K, at the diſtance of AB be next deſcribed, and through A let HL be drawn, meeting it in L; then a right-line AC drawn parallel to that joining the points L and K, will cut *de*in the point required; *as is evident from the above-mentioned problem.* By means of which the trigonometrical ſolution will alſo be brought out.

In this conſtruction, the ſolution of the problem for deſcribing a circle to touch three given circles, is determined.

## LEMMA.

*If from the Vertex of any plane Triangle* ABC, *a right-line* CD *be drawn, to divide the Baſe* AB *in the Ratio of two given Numbers* n *and* m *(ſo that* AD : BD : : n : m); *then the Sum of the Multiples of the Squares of the two Sides of the Triangle, whoſe Factors are the ſaid given Numbers, taken alternately, will be equal to the Sum of the Rectangle of the two Parts of the Baſe and the Square of the dividing Line, repeated as often as there are Units in the two propoſed Numbers (that is,* m *times* $\overline{AC}^2$+n *times* $\overline{BC}^2$=m+n *times* $\overline{AD \times BD + \overline{DC}^2}$)

For, let AD and BD be biſected in M and N, and upon AB let fall the perpendicular CE; then will $\overline{AC}^2 - \overline{DC}^2 = AD \times 2ME$ (*Elem. Cor.* 1. *to* 9. 2.) Whence it is evident that *m* times $\overline{AC}^2$ − *m* times $\overline{DC}^2$ =*m* times AD×2ME.




And, by the very ſame argument, it appears that *n* times $\overline{BC}^2$ − *n* times $\overline{DC}^2$ is = *n* times BD×2NE. Therefore,

Digitized by Google

Therefore, by adding theſe equal quantities together, $m$ times $\overline{AC}^2 + n$ times $\overline{BC}^2 - m + n$ times $\overline{DC}^2 = m$ times $AD \times 2ME + n$ times $BD \times 2NE$. But $m : n :: BD : AD$ *(by hyp.)* $:: BD \times 2NE : AD \times 2NE$ *(Elem. 7. 4.)*; and therefore $n$ times $BD \times 2NE = m$ times $AD \times 2NE$.

Let the latter of theſe equal quantities be wrote above, in the room of the former; ſo will $m$ times $\overline{AC}^2 + n$ times $\overline{BC}^2 - m + n$ times $\overline{DC}^2 = m$ times $AD \times 2ME + m$ times $AD \times 2NE = m$ times $\overline{AD \times 2ME + AD \times 2NE} = m$ times $AD \times AB = m+n$ times $AD \times BD$ (becauſe, by conſtruction $m : m+n :: BD : AB$); and conſequently $m$ times $\overline{AC}^2 + n$ times $\overline{BC}^2 = m + n$ times $\overline{DC}^2 + m + n$ times $AD \times BD$. *Q. E. D.*

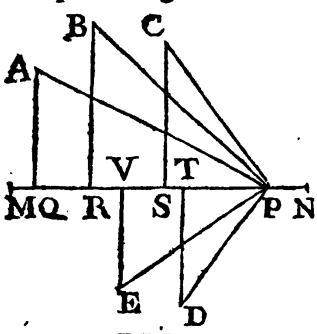
It is plain from hence that, if $m$ and $n$ be ſuppoſed to denote two given lines, inſtead of numbers, the ſum of the ſolids $m \times \overline{AC}^2$ and $n \times \overline{BC}^2$ will be equal to the ſum of the ſolids $\overline{m+n} \times \overline{DC}^2$ and $\overline{m+n} \times AD \times BD$.

PROBLEM

Digitized by Google

## PROBLEM LVIII.

*From any Number of given Points*, A, B, C, D, &c. *to draw as many Lines* AP, BP, CP, &c. *to meet in a Right-line* MN *given by Position, ſo that the Sum of all their Squares may be a given Quantity.*




Upon the given line, from all the given points, let fall perpendiculars, AQ, BR, CS, &c. which perpendiculars, as well as the diſtances between them, are all given from the poſition of MN: put, therefore, AQ $= a$, BR $= b$, CS $= c$, DT $= d$, EV $= e$, QR $= r$, QS $= s$, QT $= t$, &c. Alſo put QP $= x$, and the given ſum of all the propoſed ſquares $= k^2$

Then, RP being $= x - r$, SP $= x - s$, &c.

We ſhall have, *by Elem.* 8. 2.
$$\left.\begin{aligned} \overline{AP}|^2 &= aa + xx \\ \overline{BP}|^2 &= bb + xx - 2rx + rr \\ \overline{CP}|^2 &= cc + xx - 2sx + ss \\ \overline{DP}|^2 &= dd + xx - 2tx + tt \end{aligned}\right\} = k^2.$$

Put $aa + bb + cc + dd$ &c. $= \alpha^2$, $r + s + t$ &c. $= \beta$, and $rr + ss + tt$ &c. $= \gamma^2$; and let the number of the given points be denoted by $n$: then our equation will be reduced to $\alpha^2 + nx^2 - 2\beta x + \gamma^2 = k^2$:

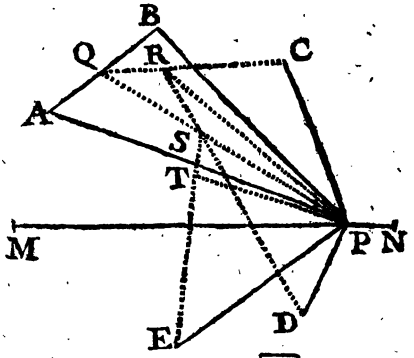
from whence $x$ is found $= \frac{\beta}{n} + \sqrt{\frac{k^2 - \alpha^2 - \gamma^2}{n} + \frac{\beta^2}{nn}}$.

*Geometrically.*

If PQ be drawn to biſect the diſtance AB, then will $\overline{AP}|^2 + \overline{BP}|^2 = 2AQ \times BQ + 2\overline{QP}|^2$, *by the lemma.* And,

Digitized by Google

And, if QC be drawn, and QR be taken equal to $\frac{1}{3}$ thereof; then will $2\overline{QP}|^2+\overline{CP}|^2=3QR\times CR$ $3\overline{PR}|^2$, *by the same.* Whence, by adding these equal quantities together, and taking $2\overline{QP}|^2$ (common) away, we have $\overline{AP}|^2+\overline{BP}|^2+\overline{CP}|^2=2AQ\times BQ+3QR$ $\times CR+3\overline{RP}|^2$. Again, by drawing RD, and taking RS = $\frac{1}{4}$ thereof, we have $3\overline{RP}|^2+\overline{DP}|^2=4RS\times DS+4\overline{SP}|^2$, *by the lemma.*




And therefore $\overline{AP}|^2+\overline{BP}|^2+\overline{CP}|^2+\overline{DP}|^2=$ $2AB\times BQ+3QR\times CR+4RS\times DS+4\overline{SP}|^2$, by the addition of equal quantities.

In the same manner, if SE be drawn, and ST be taken $=\frac{1}{5}$ thereof, we shall have $4\overline{SP}|^2+\overline{EP}|^2=5ST\times ET+5\overline{TP}|^2$. Whence, again, by the addition of equal quantities, $\overline{AP}|^2+\overline{BP}|^2+\overline{CP}|^2+\overline{DP}|^2+\overline{EP}|^2$ $=2AQ\times BQ+3QR\times CR+4RS\times DS+5ST\times ET+$ $5\overline{TP}|^2$.

Hence it is evident (without proceeding further) that, let the number of the given points be what it will, the square of the last of the lines QP, RP, SP, TP, *&c.* drawn as above, will always be a *given* quantity; because the sum of all the rectangles $2AQ\times BQ$, $3QR\times CR$, $4RS\times DS$, *&c.* (or that of their equals $AB\times QB$, $QC\times RC$, $RD\times SD$, *&c.*) is given, by the position of the points A, B, C, D, *&c.*

In order, therefore, to the geometrical conſtruction let a rectangle be conſtituted *(by Elem.* 7. 6.) equal to the exceſs of the given ſum of the ſquares above the ſum of the ſaid rectangles: then find a mean proportional between the length of the rectangle, ſo determined, and that part of its breadth defined by the number of the given points; with which, as a radius, from the laſt of the points Q, R, S, T, *&c.* let the circumference of a circle be deſcribed; which will interſect the line MN, in the point P, required.

If, inſtead of the right-line MN, the circumference of a circle (or any other curve-line) given by poſition, be propounded (wherein the lines, drawn from the given points, are required to meet), the method of conſtruction, will, it is manifeſt, be exactly the ſame.

And it may not be amiſs to obſerve farther, that if $a \times \overline{AP}|^2 + b \times \overline{BP}|^2 + c \times \overline{CP}|^2 + d \times \overline{DP}|^2$, *&c.* (where *a*, *b*, *c*, *d*, *&c.* repreſent known quantities, either numbers or lines) be ſuppoſed given, the *locus* of the point P will, *ſtill*, be the circumference of a circle, determined after the ſame manner, from the premiſed *lemma*.

For, by making in this caſe $\begin{cases} AQ : BQ :: b : a \\ QR : RC :: c : a+b \\ RS : DS :: d : a+b+c \\ ST : ET :: e : a+b+c+d, \end{cases}$ and then proceeding as above, it will appear that $a \times \overline{AP}|^2 + b \times \overline{BP}|^2 + c \times \overline{CP}|^2 +$ *&c.* is equal to $\overline{a+b} \times AQ \times QB + \overline{a+b+c} \times QR \times RC + \overline{a+b+c+d} \times RS \times SD + \overline{a+b+c+d+e} \times ST \times TE + \overline{a+b+c+d+e} \times \overline{TP}|^2$. Whence it is evident that TP is a given quantity.

PART

Digitized by Google

# PART III.

## WHEREIN

## The THEORY of GUNNERY, or the MOTION of PROJECTILES, is conſidered.

IT is, uſually, taken for granted, by *thoſe* who treat of the motion of projectiles, that the force of gravity near the earth's ſurface is every where the ſame, and acts in parallel directions; and that the effect of the air's reſiſtance upon very heavy bodies, ſuch as bombs and cannon balls, is too ſmall to be taken into conſideration.

That the error ariſing from the ſuppoſition of gravity acting uniformly, and in parallel lines, muſt be exceeding ſmall, is very obvious; becauſe, even, the greateſt diſtance of a projectile above the ſurface of the earth, is inconſiderable in compariſon of its diſtance from the center, to which the gravitation tends: but then, on the other hand, it is very certain, that the reſiſtance of the air, to very ſwift motions, is much greater than it has been commonly repreſented.

Nevertheleſs, if the amplitude of the projection, anſwering to one given elevation, be firſt found by experiment (which our method ſuppoſes to be done) the amplitudes in all other caſes, where the elevations and velocities do not very much differ from the firſt, may be determined, to a ſufficient degree of exactneſs, from the foregoing hypotheſis: becauſe, in all ſuch caſes, the effects of the reſiſtance will be nearly as the amplitudes *themſelves*; and, were they *accurately* ſo, the proportions of the



Digitized by Google

amplitudes, at different elevations, would then be the very ſame as *in vacuo*.

For this reaſon, and to avoid having recourſe to principles and calculations no ways adequate to the experience and underſtanding of beginners, for whoſe uſe this little tract is chiefly intended, I ſhall, in what follows, conform to the method of other writers, ſo far, as to take no notice of the air's reſiſtance; but conſider the motions as performed *in vacuo*.

Now, in order to form a clear idea of the ſubject here propoſed, the path of every projectile is to be conſidered as depending on two different forces; that is to ſay, on the impellant force, whereby the motion is firſt began (and would be continued, in a right-line), and on the force of gravity, by which the projectile, during the whole time of its flight, is continually urged downwards, and made to deviate more and more from its firſt direction.

As whatever relates to the track and flight of a ball (neglecting the reſiſtance of the air) is to be determined from the action of theſe two forces, it will be proper, before we proceed to conſider their joint effect, to premiſe ſomething concerning the nature of the motion produced by each, when ſuppoſed to act alone, independant of the other; to which end the two firſt, of the four following *lemmas*, are premiſed.

LEMMA

Digitized by Google

## LEMMA I.

*Every Body, after the impreſſed Force, whereby it is put in Motion, ceaſes to act, continues to move uniformly in a Right-line; unleſs it be interrupted by ſome other Force or Impediment.*

This is a law of nature, and has its demonſtration from experience and matter of fact.

### COROLLARY.

It follows from hence that a ball, after leaving the mouth of the piece, would continue to move along the line of its firſt direction, and deſcribe ſpaces therein proportional to the times of their deſcription, were it not for the action of gravity; whereby the direction is changed, and the motion interrupted.

## LEMMA II.

*The Motion, or Velocity, acquired by a Ball, in freely deſcending from Reſt, by the Force of an uniform Gravity, is as the Time of the Deſcent; and the Space fallen through, as the Square of that Time.*

The firſt part of the *lemma* is extremely obvious: for, ſince every motion is proportional to the force whereby it is generated, that generated by the force of an uniform gravity muſt be as the time of the deſcent; becauſe the *whole effort* of ſuch a force is proportional to the time of its action; that is, as the time of the deſcent.



Digitized by Google

To demonſtrate that the diſtances deſcended are proportional to the ſquares of the times, let the time of falling through any propoſed diſtance AB be repreſented by the right-line PQ; which conceive to be divided into an indefinite number of very ſmall, equal, particles, repreſented, each, by the *ſymbol m*; and let the diſtance deſcended in the firſt of them be Ac; in the ſecond *cd*; in the third *de*; and ſo on.

Then the velocity acquired being always as the time from the beginning of the deſcent, it will at the middle of the firſt of the ſaid particles be repreſented by $\frac{1}{2}m$; at the middle of the ſecond, by $1\frac{1}{2}m$; at the middle of the third, by $2\frac{1}{2}m$, *&c.* Which values conſtitute the ſeries $\frac{m}{2}$, $\frac{3m}{2}$, $\frac{5m}{2}$, $\frac{7m}{2}$, $\frac{9m}{2}$, *&c.*

But, ſince the velocity, at the middle of any one of the ſaid particles of time, is an exact mean between the velocities at the two extremes thereof, the correſponding particle of the diſtance AB may be therefore conſidered as deſcribed with that mean velocity: and ſo, the ſpaces Ac, *cd*, *de*, *ef*, *&c.* being, reſpectively, equal to the abovementioned quantities $\frac{m}{2}$, $\frac{3m}{2}$, $\frac{5m}{2}$, $\frac{7m}{2}$, *&c.* it follows, by the continual addition of theſe, that the ſpaces Ac, Ad, Ae, Af, *&c.* fallen through from the beginning, will be expreſſed by $\frac{m}{2}$, $\frac{4m}{2}$, $\frac{9m}{2}$, $\frac{16m}{2}$, $\frac{25m}{2}$, *&c.* Which are, evidently, to one another in proportion, as 1, 4, 9, 16, 25, *&c.* that is, as the ſquares of the times. *Q. E. D.*

COROL.

Digitized by Google

### COROLLARY.

Seeing the velocity acquired in any number *(n)* of the aforeſaid, equal, particles of time (meaſured by the ſpace that would be deſcribed in one ſingle particle) is repreſented by $n$ times $m$, or $nm$, it will therefore be, as 1 particle of time, is to $n$ ſuch particles, ſo is $nm$, the ſaid diſtance anſwering to the former time, to the diſtance, $n^2m$, correſponding to the latter, with the ſame celerity, acquired at the end of the ſaid $n$ particles. Whence it appears that the ſpace $\frac{n^2m}{2}$ *(found above)* through which the ball falls, in any given time $n$, is juſt the half of that $(n^2m)$ which might be uniformly deſcribed with the laſt, or greateſt, celerity, in the ſame time.

### SCHOLIUM.

It is found, by experiment, that any heavy body, near the earth's ſurface (where the force of gravity may be conſidered as uniform) deſcends about 16 feet, from reſt, in the firſt ſecond of time.

Therefore, as the diſtances fallen through are proved above to be, in proportion, as the ſquares of the times; it follows that, as the ſquare of 1 ſecond, is to the ſquare of any given number of ſeconds, ſo is 16 feet, to the number of feet a heavy body will freely deſcend in the ſaid given number of ſeconds. Whence the number of feet deſcended in any given time will be found, by multiplying the ſquare of the number of ſeconds by 16.

Thus the diſtance deſcended in 2, 3, 4, 5, *&c.* ſeconds, will appear to be 64, 144, 256, 400 F, *&c.* reſpectively.

Moreover, from hence, the time of the deſcent through any given diſtance will be obtained, by dividing the ſaid diſtance, in feet, by 16, and extracting the ſquare root of the quotient; or, which comes to the ſame

Digitized by Google

ſame thing, by extracting the ſquare root of the whole diſtance, and then taking $\frac{1}{4}$ of that root for the number of ſeconds required. Thus, if the diſtance be ſuppoſed 2640 feet; then, by either of the two ways, the time of the deſcent will come out 12.84 ſeconds, or $12''.\ 50'''.\ 4''''$.

It appears alſo *(from the corol.)* that the velocity *per* ſecond (in feet), at the end of the fall, will be determined, by multiplying the number of ſeconds in the fall by 32: thus it is found that a ball, at the end of 10 ſeconds, has acquired a velocity of 320 feet *per* ſecond.——After the ſame manner, by having any two of the four following quantities, *viz.* the *force*, the *time*, the *velocity*, and *diſtance*, the other two may be determined: but this not being abſolutely neceſſary in what follows (though equally uſeful in other diſquiſitions) I ſhall put down the ſeveral rules, or equations below, in a note,* to be taken, or omitted, at pleaſure.

LEMMA

---

* *Let the ſpace freely deſcended by a ball, in the firſt ſecond of time (which is as the accelerating force) be denoted by* f, *alſo let* t *denote the number of ſeconds wherein any diſtance,* d, *is deſcended; and let* v *be the velocity,* per *ſecond, at the end of the deſcent: then will*

$$v = 2ft = 2\sqrt{fd} = \frac{2d}{t}$$

$$t = \sqrt{\frac{d}{f}} = \frac{v}{2f} = \frac{2d}{v}$$

$$d = ftt = \frac{vv}{4f} = \frac{tv}{2}$$

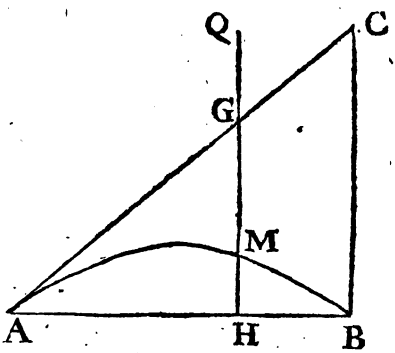
$$f = \frac{d}{tt} = \frac{v}{2t} = \frac{vv}{4d}.$$

*All which equations are very eaſily deduced from the two original ones,* $d = ftt$, *and* $v = 2ft$, *already demonſtrated; the former in the propoſition itſelf; and the latter, in the corollary to it; by which it appears that the meaſure*

## LEMMA III.

*A Ball, projected in the Direction of a Right-line* AC *making an Angle with the Horizon* AB, *is, by Means of its Gravity, deflected continually, more and more, from its firſt Direction; but the Celerity with which it approaches any Perpendicular* (BC) *to the Horizon, is neither increaſed nor decreaſed by the Action of Gravity.*

For, let a line HMQ, perpendicular to the horizon, be conceived to move parallel to itſelf, towards BC, along with the ball: then, as the gravity always acts in this line, it can have no effect in making it, either, move faſter or ſlower towards BC; but is wholly employed in drawing down the projectile along the ſame, from its firſt direction AC; and thereby cauſing it to deſcribe a curve-line AMB.




COROL.

---

*meaſure of the velocity at the end of the firſt ſecond is* 2f; *whence the velocity* (v) *at the end of* t *ſeconds muſt conſequently be expreſſed by* 2f × t *or* 2ft.

*Having proceeded thus far, I ſhall here take the opportunity to point out, and rectify an inadvertant expreſſion, at p.* 230 *of my Book of Fluxions, relating to this ſubject.* —*It is there ſaid, by way of remark,* that, whatever ratio the times have with reſpect to the diſtances deſcended, *&c.* the ſame alſo will the velocities have,



Digitized by Google

### COROLLARY.

Hence it appears that the projectile, at the end of a given time, is in the very ſame vertical line HG, as it would be in, if gravity was not to act; and that the horizontal diſtance AH, as well as the hypothenuſe AG, is proportional to the time wherein the projectile actually moves through the arch AM, correſponding to the ſaid diſtance.

---

being proportional to the times. *Which obſervation, it is clear from the words themſelves, ought to be reſtrained to thoſe caſes, where the velocities are proportional to the times; that is, where the accelerating forces are equal. Therefore, inſtead of the ſaid obſervation, as it now ſtands, read,* What is above demonſtrated with reſpect to the times, holds alſo in the velocities, when the accelerating forces are equal.

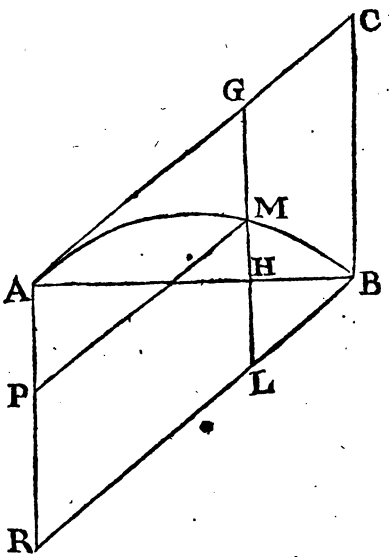
LEMMA

Digitized by Google

## LEMMA IV.

*The Deflections arising from the Action of Gravity, or the vertical Distances,* MG, BC, *intercepted by the Path of the Ball and the Line of Direction, are in proportion, to one another, as the Squares of the corresponding Parts* AG, *and* AC, *of the said Line of Direction.*

Conceive a line GML to be carried along, parallel to BC, so that its extreme point G may trace out the line of direction AC, in the very same manner as the projectile itself, would describe it, were it not to be deflected therefrom by the action of gravity: then, since, by the preceding lemma, the projectile is always in the line GML, and the force of gravity is, wholly, employed in urging it downwards, along that line; the effect produced by the said force, or the distance MG of the ball from the extreme point G, at the end of any given time, will consequently be the very same, as if the line GML (instead of being carried uniformly towards CB) was to have continued in its first position AR, and the ball suffered to descend from rest along that line; the force employed being the same in both cases. But it is proved, *in Lemma* 2, that the spaces AP and AR that would be described in descending, freely,





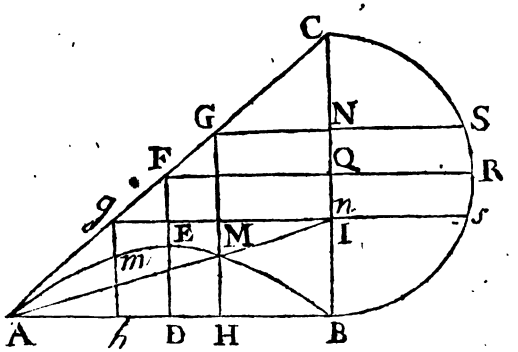

Digitized by Google

from reſt, are as the ſquares of the times; therefore their equals GM and CB are likewiſe as the ſquares of the ſame times; that is, of the times wherein AM and AMB are deſcribed: which are to each other as $\overline{AG}|^2$ to $\overline{AC}|^2$ *(by Corol. to Lem.* 3.)

### COROLLARY I.

If B be taken as the point where the projectile impinges on the horizon AB, and in BC (perpendicular to AB) there be taken CI=GH; then a right-line, drawn from I to A, will cut the vertical line HG in the very point (M) through which the center of the projectile paſſeth.

For $\overline{AC}|^2 : \overline{AG}|^2 :: BC : MG$, *as above.*
and $\overline{AC}|^2 : \overline{AG}|^2 :: \overline{BC}|^2 : \overline{HG}|^2$, *by ſim. triangles.*
Therefore. $BC : MG :: \overline{BC}|^2 : \overline{HG}|^2$, *by equality of ratios:*
and conſequently $MG \times BC = \overline{HG}|^2$.




This, turned into an analogy, gives BC : HG :: HG : MG; and ſo likewiſe is BC to CI (becauſe of the parallel lines.) Whence it is evident that HG and CI are equal to each other, in all poſitions of HG: by means whereof, as many points in the curve, as you pleaſe, may be determined.

COROL.

Digitized by Google

COROLLARY II.

If GNS be drawn, parallel to AB, interfecting BC in N, and meeting the circumference of a femi-circle, defcribed upon BC, in S; it will further appear, that the height (HM) of the ball is everywhere a third proportional to BC and NS. For, fince BC : HG : : HG ; MG, it follows, by divifion, that BC : CN : : HG (BN) : HM $= \frac{CN \times BN}{BC} = \frac{\overline{NS}^2}{BC}$ *(Elem.* 16. 3.) Hence it is evident that the projectile will be at its greateft height, when it has performed, juft, the half of its flight; fince it is well known that NS (and confequently $\frac{\overline{NS}^2}{BC}$) is the greateft poffible when it coincides with the radius QR.

It appears moreover that the heights *hm*, HM, of the ball, in any two vertical lines, equally diftant from that paffing through the higheft point E of the trajectory, are equal; becaufe the correfponding ordinates *ns* and NS are equal, as being equally diftant from the center Q of the femi-circle.—Laftly, it is apparent that the greateft altitude DE is $= \frac{1}{4}$BC; becaufe $\frac{\overline{NS}^2}{BC}$, when NS coincides with QR, becomes $= \frac{\overline{\frac{1}{2}BC}^2}{BC} = \frac{1}{4}$ BC. Which may be otherwife made out, by confidering, that, as AF is but the half of AC, its fquare will be only $\frac{1}{4}$ of the fquare of AC; and therefore FE only $\frac{1}{4}$ of BC *(by the lem.)* And fo, DF being $= \frac{1}{2}$BC *(by fim. triang.)* DE, as well as FE, muft be $= \frac{1}{4}$BC.

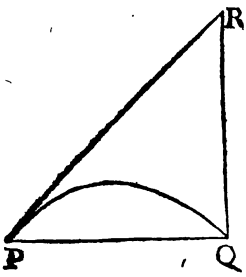
PROPOSITION

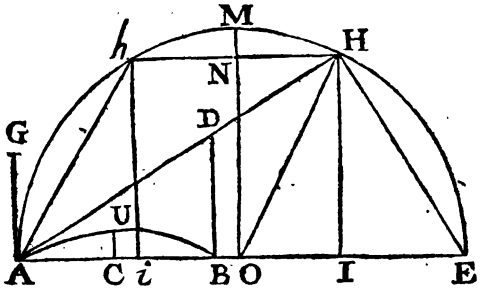
Digitized by Google

## PROPOSITION I.

*The Amplitude, or horizontal Range of a Piece, with a given Charge of Powder, at an Elevation of* 45 *Degrees, being known, from Experiment; to determine the Elevation ſo as to hit an Object, at a given Diſtance, on the plane of the Horizon; the Quantity of Powder remaining the ſame.*

Let PQ be the given amplitude, at an elevation (QPR) of 45 degrees; let AB be the given diſtance of the object, and BAD the required elevation: in AB, produced, take AO=PQ, with which as a radius, from the center O, let a ſemi-circle AME be deſcribed, and let AD, produced, meet the periphery thereof in H; join E, H, and make DB and HI perpendicular to AE.




Since the charge of powder, or the velocity at both the elevations QPR and BAD, is ſuppoſed to be the ſame, the times of flight, during which the diſtances PR and AD would be uniformly deſcribed with that velocity, will therefore be to each other, directly as the ſaid diſtances; and conſequently $\overline{PR}^2 : \overline{AD}^2 :: RQ$

Digitized by Google

RQ : DB *(by principles already explained, vid. Lem.* 3 *and* 4.)

But $\overline{PR}^2$ ($=\overline{PQ}^2+\overline{RQ}^2$) $=2\overline{RQ}^2$ (becauſe the angle P ($=45°$) $=$R.) Hence the above proportion becomes $2\overline{RQ}^2 : \overline{AD}^2 :: RQ : DB$; from which we have $\overline{AD}^2=2RQ\times DB=AE\times DB$, *by conſtruction.*

Moreover, from the ſimilar triangles ABD and AHE, we have AE : EH : : AD : DB; whence EH $\times$ AD $=$ AE $\times$ DB $=\overline{AD}^2$ *(p. above)* and conſequently EH $=$ AD: and ſo, the triangles EHI and ADB being equiangular, it is plain that HI is alſo equal to AB. Whence follows this eaſy

*Conſtruction.*

With an interval equal to the given amplitude, at the elevation of 45°, let a ſemi-circle AME be deſcribed; make OM perpendicular to the diameter thereof, in which take ON equal to the given diſtance AB, and through N, parallel to AE, draw HN*h*, interſecting the circumference in H and *h*; then either of the directions AH, or A*h*, will anſwer the conditions of the problem. *Q. E. I.*

COROLLARY I.

If OH be drawn, the angle EOH will be $=2$EAH *(Elem.* 9. 3.) and it will be, as OH (PQ) : HI (AB) : : *radius* : *ſine* EOH; that is, in words, as the given amplitude, at 45° elevation, is to any other propoſed amplitude, ſo is the *radius*, to the *ſine* of twice the elevation correſponding to the latter. From whence it is evident, that the horizontal amplitudes, at different elevations, are to one another as the *ſines* of the doubles of the ſaid elevations; and that, the amplitude of the projection at an elevation of 45 degrees (when HI coincides with MO) is the greateſt poſſible.

COROL.

Digitized by Google

### COROLLARY II.

Since it is found above that $AE \times DB\ (=\overline{AD}^2) = \overline{EH}^2$, it follows, that $\overline{AE}^2 : \overline{EH}^2 :: \overline{AE}^2 : AE \times DB :: AE : DB :: \frac{1}{2}AO\ (\frac{1}{4}AE) : CV\ (\frac{1}{4}DB$. *vid. Corol. 2. to Lem. 4.*) that is, as the ſquare of the *radius*, is to the ſquare of the *ſine* of the angle of elevation, ſo is half the greateſt horizontal amplitude, to the greateſt altitude of the projectile.

Hence it appears, that the diſtance which the ball would aſcend, if projected in a vertical direction *(uſually called the impetus)* is juſt one half of the greateſt amplitude; ſince, in this caſe, the *ſine* of the elevation becomes equal to the *radius*.

Therefore, as a body *(in vacuo)* aſcends and deſcends with the ſame celerity; and ſeeing the diſtance AG, expreſſing the perpendicular aſcent, is as the ſquare of the celerity at A *(p. Lem. 2)*; it follows that the greateſt horizontal amplitude AO, being $=2AG$, is alſo as the ſquare of the ſame celerity.

From whence, and *Corol. 1*, it is manifeſt, that the amplitudes, when both the elevations, and the velocities, differ, will be to each other in a *ratio* compounded of the *ratio* of the *ſines* of the double elevations, and the *duplicate ratio* of the velocities.

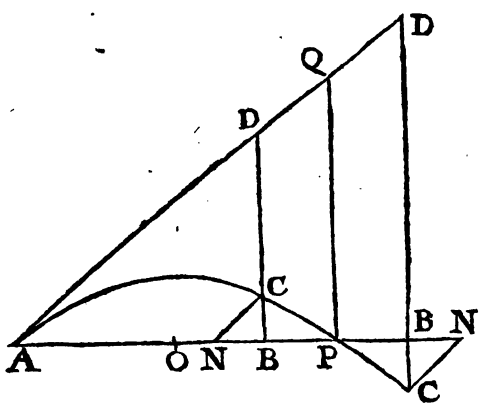
PROPOSITION

Digitized by Google

## PROPOSITION II.

*The greateſt horizontal Amplitude, and the Angle of Elevation being given; to find at what Diſtance the Piece ought to be planted, to hit an Object, whoſe Height above, or Depreſſion below, the Level of the Piece is alſo given.*

Let BC be the perpendicular height, or depreſſion of the propoſed object, and AB the required diſtance; let BC, produced, meet the line of direction AD in D, and let P be the place where the path of the projectile (produced) meets the level of the piece: make PQ perpendicular to AP, and CN parallel to AD.




By the laſt propoſition it will be, as *radius* : *ſin.* 2A : : the greateſt horizontal amplitude, to the diſtance AP; which, therefore, is known.

Moreover, it appears from the fourth proportion, *in Corol.* 1, *to Lem.* 4, that PQ : BD : : BD : CD.

But $\left\{\begin{matrix} PQ : BD :: AP : AB \\ BD : CD :: AB : AN \end{matrix}\right\}$ *by ſimilar triangles.*

Therefore, *by equality of ratios.*

AP : AB : : AB : AN;



Digitized by Google

whence AP : PB : : AB : BN *(by diviſion)*
and conſequently $AP \times BN = AB \times PB$.

Let AP be now biſected, in O; then $AB \times BP$ (or its equal $AP \times BN$) being $= \overline{AO}|^2$ ~ $\overline{OB}|^2$ *(Elem.* 7. 2.) we therefore have $\overline{OB}|^2 = \overline{AO}|^2 \mp AP \times BN = AO \times \overline{AO \mp 2BN}$; whence AB is likewiſe known. *Q. E. I.*

### COROLLARY.

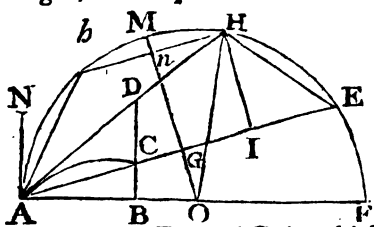
If the amplitude AP, the elevation PAQ, and the diſtance AB of any perpendicular BCD from the place of projection A, be ſuppoſed given; then the height, or depreſſion of the ball in that perpendicular, may, from hence, be found.

For it is proved that AP : BP : : AB : BN; from which BN is given: but, as the *radius*, is to the *tangent* of BNC (or BAD) ſo is BN to BC.

## PROPOSITION III.

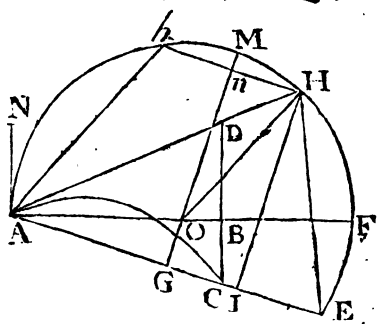
*Having the greateſt horizontal Amplitude, or Range of a Piece, with the Diſtance and the Height (or Depreſſion) of the Object, to find the Angle of Elevation.*

Let BC *(in fig.* 1 *and* 2) be the perpendicular height, or depreſſion of the object, AB the given horizontal diſtance, and AH the required direction: alſo let PQ *(fig.* 3.) be the greateſt horizontal amplitude, anſwering to 45° of elevation, *(vid. Corol.* 1. *Prop.* 1.) Draw AC, in which produced (if need be) take AG = PQ; make MGO perpendicular to AG, meeting AB produced (if neceſſary) in O; and from the center O, with the interval OA, let the circumference of a circle be deſcribed, interſecting AG produced,

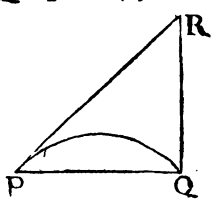



Digitized by Google

duced, in E; join E, H, and let HI, AN, and QR, be perpendicular to AE, AO, and PQ reſpectively, and let BC be produced to meet AH in D.




It appears, from what has been already delivered, that $\overline{AD}|^2 : \overline{PR}|^2 :: DC : RQ$. Whence, $\overline{PR}|^2$ being $= 2\overline{PQ}|^2 = 2\overline{AG}|^2 = \frac{1}{4}\overline{AE}|^2$, and $RQ = PQ = \frac{1}{2}AE$ *(by conſtruction)* we have $\overline{AD}|^2 : \frac{1}{2}\overline{AE}|^2 :: DC : \frac{1}{2}AE$; and conſequently $\overline{AD}|^2 = AE \times DC$.




But, the triangles ADC and AEH being equiangular (becauſe ADC = DAN = E, and DAC common) we likewiſe have AD : DC :: AE : EH; and therefore $AE \times DC = AD \times EH = \overline{AD}|^2$ *(p. above)* whence EH = AD; and ſo, the triangles ADB and EHI being equiangular, we likewiſe have HI = AB; and from thence this eaſy

*Conſtruction.*

Having deſcribed the circle AHEF, as above directed, and drawn MG perpendicular to AE, take therein G*n* equal to the given horizontal diſtance AB, and through *n*, parallel to AE, draw H*h*, cutting the circumference of the circle in H and *h*; then either of the directions AH, or A*h*, will anſwer the conditions of the problem.

From this conſtruction we have the following calculation, *viz.* As AB : BC :: AG (PQ) : GO; which, added to, or ſubtracted from G*n* (AB), gives O*n*.

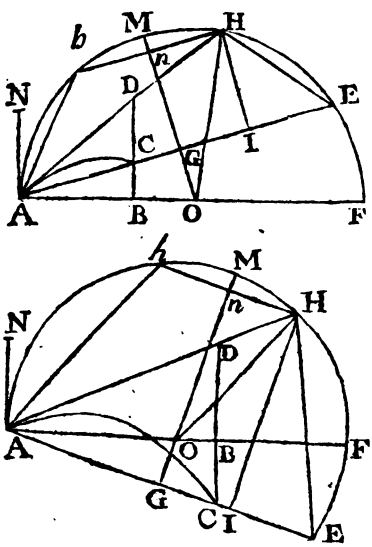


Digitized by Google

Then, as AG (PQ) : *On* : : *co-ſine* of OAG : *co-ſine* HO = (H*Ah*) the difference of the two required elevations: from which the elevations *themſelves* will be known: and from whence, if BC be ſuppoſed to vaniſh, the ſolution to *Prop.* 1. will be obtained; being only a particular caſe of that above. *Q. E. I.*

## COROLLARY I.

Hence the time of the flight may likewiſe be determined: for, *S.* AHE (ACD) : *S.* HAE : : AE : $\left(\frac{S.HAE}{S.AHE} \times AE =\right)$ HE = AD *(p. above)* which being proportional to the time of the flight *(vid. Lem.* 3.) it follows that the ſaid time will, always, be as $\frac{S.HAE}{S.ACD}$; becauſe AE is ſuppoſed conſtant. Therefore, if the time of the perpendicular aſcent, or deſcent, through the impetus AN, be found, and denoted by $T$, it is evident that the time ſought will be truly repreſented by $\frac{S.HAE}{S.ACD} \times 2T$; that is, the *co-ſine* of the object's elevation above, or depreſſion below the level of the piece, will be to the *ſine* of the elevation of the piece above the object, as twice the time of the perpendicular aſcent, or deſcent, is to the true time of the flight.




COROL.

Digitized by Google

### COROLLARY II.

If the elevation of the piece, together with the diſtance, and the height (or depreſſion) of the object, be given, the *impetus* may, from hence, be alſo found.

For, firſt, it will be AB : BC : : *radius* : *tang.* BAC; whence (as BAD is given) EAH will likewiſe be known.

Then, *S.* EAH (CAD) : *S.* AHE (ACD) : : HE (AD) : AE. Alſo *S.* ADC : *radius* : : AB : AD.

Therefore, by compounding theſe proportions, we have *S.* CAD × *S.* ADC : *radius* × *S.* ACD : : AB : AE; which is four times the required impetus (*vid. Corol.* 3. *to Prop.* 1.)

### COROLLARY III.

Moreover, if the elevation, and the impetus be given, the amplitude of the projection on an aſcending, or deſcending plane ACE, whoſe inclination is given, may from hence be derived.

For, *S.* AHE (ACD) : *S.* EAH (CAD) : : AE : EH (AD.) And S. ACD : *S.* ADC : : AD : AC.

By the compoſition of which proportions we have □ *S.* ACD : *S.* CAD × *S.* ADC : : AE : AC; whence AC is given.

### COROLLARY IV.

Hence, alſo, may the ratio of the amplitudes, on the ſame plane, at different elevations, be deduced; for the firſt and third terms, of the laſt proportion, continuing invariable, the ratio of the 2$^{d}$ and 4$^{th}$ will likewiſe be invariable; that is, the rectangle under the *ſine* of the elevation above the plane, and the *co-ſine* of the elevation above the horizon, in any one caſe, will be to the like rectangle, in any other, as the amplitude in the former caſe, to the amplitude in the latter.

COROL.

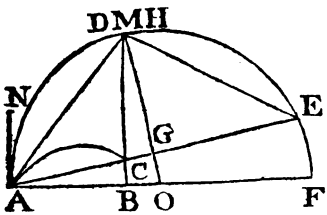
Digitized by Google

### COROLLARY V.

But, if the elevation be ſuppoſed conſtant, and the plane's inclination to vary; then, ſince, by the above proportion, AC is *univerſally* $= \frac{S.\,CAD \times S.\,ADC}{\square\ fine\ ACD} \times$ AE (where AE and the angle ADC are ſuppoſed conſtant) it follows that the amplitude will be, barely, as $\frac{S.\,CAD}{\square\ S.\,ACD}$; that is, *inverſely*, as the ſquare of the *co-ſine* of the inclination of the plane, applied to the *ſine* of the elevation above the plane.—If both the inclinations, and the elevations, differ, it will appear, from the ſame equation, that, the amplitude will be, *univerſally*, as the rectangle of the *ſine* of the elevation above the plane, and the *co-ſine* of the elevation above the horizon, applied to the ſquare of the *co-ſine* of the plane's inclination.

### COROLLARY VI.

Since it is proved that HI is, always, equal to AB, it is evident that, when the former coincides with MG, and thereby becomes a *maximum*, the latter will alſo be a *maximum*: in which circumſtance AC will likewiſe be a *maximum*; and the point D will then coincide with M and H *(as in the annexed figures)* becauſe AD and EH are always equal to each other.



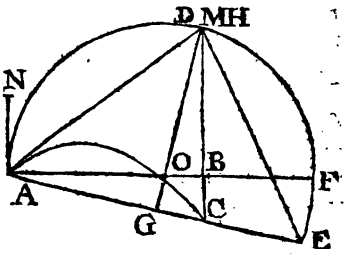

Therefore, ſince, in this caſe, the angle HAE is (= E) = NAH, it appears that the amplitude, on any inclined plane, will be the greateſt poſſible when the line of direction AH biſects the angle EAN, included between the plane and the zenith.

COROL.

Digitized by Google

### COROLLARY VII.

Hence the greateſt amplitude, on any inclined plane, with a given impetus, may be determined: for the right-angled triangles AOG and HOB, having AO = HO, and the angle O, common, are equal in all reſpects: therefore it will be, as the *tangent* of AHG (or BAH, the piece's elevation) is to the *tang.* of CHG (or CAB, the plane's inclination) ſo is AG (twice the given impetus) to the difference CG between the ſaid double impetus and the amplitude ſought.




### COROLLARY VIII.

Hence, alſo, if the greateſt amplitude, on an inclined plane be given, the impetus may be found: for, it will be as the *radius*, is to the *ſine* of the plane's inclination BAC, ſo is the given amplitude AC to the difference (BC or CG) betwixt it and twice the impetus. *Vid. Corol.* 2. *to Prop.* 1.

### COROLLARY IX.

But if, inſtead of the plane's inclination, the perpendicular height, or depreſſion of the object above, or below the level of the piece, be given; then, AC being $= AG \mp BC$, and $\overline{AB}^2$ $(= \overline{AC}^2 - \overline{BC}^2) = \overline{AG}^2 \mp 2AG \times BC$, the greateſt diſtance AB, at which the ball can poſſibly hit the object, will therefore be $= \sqrt{AG \times \overline{AG \mp 2BC}}$. From which, as all the ſides of the triangle ABC are given, the angle BAC

BAC will likewiſe be known; and from thence, the elevation, BAH, *by Corol.* 6.

From the latter of the two caſes here conſidered (where the object is ſuppoſed below the level of the piece, *as in fig.* 2.) the greateſt amplitude of a ball, projected from a given height above the plane of the horizon, is given; being $= \sqrt{AG \times \overline{AG + 2BC}}$, *(as above)* in which BC denotes the given height.

## COROLLARY X.

But, if the horizontal diſtance AB is given, and it be required to determine the greateſt height the ball can poſſibly reach, in the perpendicular BCD; we ſhall then have, HG (AB) : AG : : *radius* : *tang.* of the elevation (BAH or AHG); and, as *radius* : *tang.* BAC ($=2BAH \sim 90$) : : AB : BC; which therefore is known. But, becauſe $AC = AG + BC$, and $\overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2$, the value of BC will, *alſo*, be truly expreſſed by $\frac{AG^2 \sim AB^2}{2AG}$.

## CORALLARY XI.

Laſtly, if there be given the perpendicular height or depreſſion, of the object, and its horizontal diſtance, in order to determine the elevation, and the *leaſt* impetus, to hit the object: then it will be as AB : BC : : *radius* : *tang.* BAC; whence the elevation BAH is alſo known, *by Corol.* 6: and, as *radius* : *tang.* AHG (BAH) : : HG (AB) : AG; the half of which is the impetus, *by Prop.* 1. *Corol.* 2.

*Here*

Digitized by Google

## *Here follow the practical Solutions of the ſeveral Caſes depending on the foregoing* THEORY.

### I. *Of Projections made on the Plane of the Horizon.*

### PROBLEM I.

*The greateſt Amplitude of a Piece being known (from Experiment) to find the Amplitude, at any propoſed Elevation.*

SOLUTION.

As the radius is to the ſine of double the propoſed elevation, ſo is the given, to the required, amplitude *(by Prop.* 1. *Cor.* 1.)

Ex. Let the gr. amp. be 8000 feet, and the given elev. 30° 16′.

| | | |
|---|---|---|
| Then, as radius | . . . | 10. 0000 |
| to ſine 60° 32′ | . . . | 9. 9398 |
| ſo is 8000 | . . . | 3. 9030 |
| Amp. req. 6965 f. | . . . | 3. 8428 |

### PROBLEM II.

*The Impetus, or the greateſt Amplitude (which is the Double thereof) being known, to find the Elevation, to ſtrike an Object at a given Diſtance.*

SOLUTION.

As the greateſt amplitude, is to the given diſtance; ſo is the radius, to the ſine of the double elevation *(by Prop.* 1. *Corol.* 1.)

Ex. Let the gr. amp. be 7500 f. and the given diſtance 5620 f.



Digitized by Google

| | |
|---|---|
| Then, as 7500 . . . | 3. 8750 |
| is to 5620 . . . | 3. 7497 |
| ſo is radius . . . | 10. 0000 |
| to the ſine of 48° 32′, or 131° 28′ | 9. 8747 |

Therefore 24° 16′ is the lower, and 65° 44′ the higher elevation, required.

## PROBLEM III.

*The Angle of Elevation, and the Diſtance of an Object, being given, to find the Impetus, ſo as to ſtrike the Object.*

### SOLUTION.

As the ſine of twice the elevation, is to the radius, ſo is the diſtance of the object to twice the impetus *(by Prop. 1.)*

Ex. Let the elev. be 32° 12′ and the given diſt. 6500 f.

| | |
|---|---|
| Then, as ſine 64° 24′ . . . | 9. 9551 |
| is to radius . . . | 10. 0000 |
| ſo is . 6500 . . . | 3. 8129 |
| to . . 7208 . . . | 3. 8578 |

Whence 3604 is the impetus required.

## PROBLEM. IV.

*The Amplitude, at any one known Elevation being given, to find the Amplitude at any other known Elevation.*

### SOLUTION.

As the ſine of double the firſt elevation, is to the ſine of double the ſecond, ſo is the amplitude at the former, to that at the latter *(by Prop. 1.)*

Ex. Let the firſt elev. be 25° 12′, the ſecond 36° 15′, and the given amp. 5250 f.

Then,

Digitized by Google

| | | | |
|---|---|---|---|
| Then, as sine 50° 24′ | . | co. ar. | 0. 1132 |
| to sine . 72° 30′ | . | . | 9. 9794 |
| so is . . 5250 f. | . | . | 3. 7201 |
| to the amp. req. 6498 f. | . | . | 13. 8127 |

## PROBLEM V.

*The Amplitude of the Projection, with a given Quantity of Powder, being known; to find the Requisite of Powder, so as to strike an Object at a given Distance; the Elevation remaining the same.*

### SOLUTION.

As the given amplitude, is to the proposed distance, so is the given weight, or quantity of powder, to the quantity sought, nearly. *

Ex. Suppose the requisite of powder to throw a shot 4000 feet, at 45° elevation, to be 16lb. What quantity is necessary, to strike an object at the distance of 5000 feet.

Here it will be, as 4000 : 5000 : : 16 : 20lb. the quantity sought.

---

* *In this solution, the velocity communicated to the ball, is supposed to be in the subduplicate ratio of the quantity of powder; which is not strictly true, especially in large charges; since a considerable part of the powder, in such cases, is blown out, unfired: there are, besides, other reasons to be assigned, why, the velocity cannot be exactly in the proportion above specified.*



Digitized by Google

## PROBLEM VI.

*The Diſtance of the Object, and the Elevation of the Piece, being given; to determine the Time of the Flight.*

### SOLUTION.

As the radius is to the tangent of the elevation, ſo is the given diſtance of the object, *in feet*, to the ſquare of four times the number of *ſeconds*, required *(vid. p.* 184 *and* 187.)

Ex. Let the elevation be 32°, and the diſtance of the object 5280 feet.

| | |
|---|---|
| Then, radius . . . | 10. 0000 |
| tang. 32° . . . | 9. 7958 |
| diſt. 5280 . . . | 3. 7226 |
| | 13. 5184 |
| 57° 44 . . . . | 1. 7592 |

¼ of which, or 14',36, is the time required.

But, when the elevation is 45 degrees (which is commonly the caſe, in throwing bombs, *&c.)* then ¼ of the ſquare root of the diſtance, *in feet*, will give the number of ſeconds taken up in the flight. The knowing of which will be of uſe in adjuſting the fuſe.

## II. *Of Projections, when the Object fired at, is above, or below, the Level of the Piece.*

## PROBLEM VII.

*The horizontal Diſtance, and the Angle of Elevation, or Depreſſion of an Object, being given, with the Elevation of the Piece; to find the Impetus, ſo as to hit the Object.*

### SOLUTION.

As the rectangle of the ſine of the elevation of the piece above the object, and the co-ſine of its elevation

Digitized by Google

vation above the horizon, is to the rectangle under the radius and the co-ſine of the object's elevation, or depreſſion; ſo is ½ of the given horizontal diſtance of the object, to the impetus required *(by Prop* 3. *Corol.* 2.)

Ex. Let the horizontal diſtance of the object be 5600 feet, and its elevation 8° 15′; and let the elevation of the piece be 32° 30′: then

| | | |
|---|---|---|
| 24° 15′, co-ar. of its ſine . . | 0. 3865 | to be added |
| 32° 30′, co-ar. of its co-ſ. . | 0. 0740 | |
| radius . . . . . | 10. 0000 | |
| 8° 15′, its co-ſine . . . | 9. 9955 | |
| 1400 f. . . . . | 3. 1461 | |
| 4000 f. the imp. req. . . | 2[3. 6021 | |

## PROBLEM VIII.

*The horizontal Diſtance, and the Angle of Elevation, or Depreſſion of an Object, being given, together with the Impetus; to find the Elevation of the Piece, to hit the Object.*

### SOLUTION.

As the radius is to the tangent of the object's elevation, or depreſſion, ſo is twice the impetus to a fourth number; which add to, or ſubtract from, the given horizontal diſtance, according as the object is elevated, or depreſſed: then ſay,

As twice the impetus is to the ſum, or remainder, ſo is the co-ſine of the given elevation, or depreſſion, to the co-ſine of an angle; which added to, and ſubtracted from, the angle included between the object and the zenith (or vertical point), gives the double of the complements of two different elevations, whereby the ball may hit the object *(ſee Prop.* 3.)

Ex. Let the horizontal diſtance of the object be 5600 feet, and its elevation 8° 15′; and let the given impetus be 4000 f.

Then,

Digitized by Google

| | |
|---|---|
| Then, as radius . . . . | 10. 0000 |
| is to tang. 8° 15′ . . . | 9. 1613 |
| ſo is 8000 . . . . | 3. 9031 |
| to . 1160 . . . . | 13. 0644 |

This added to 5600 gives 6760.

| | |
|---|---|
| Therefore, as 8000 . . . | 3. 9031 |
| is to . 6760 . . . | 3. 8299 |
| ſo is co-ſ. 8° 15′ . . . | 9. 9955 |
| to the co-ſ. 33. 15 . . . | 9. 9223 |

Which, added to, and ſubtracted from 81° 45′, gives 115° 00′, and 48° 30′, reſpectively: the halves of which are 57° 30′, and 24° 15′; whoſe complements 32° 30′, and 65° 45′, are the two elevations required.

## PROBLEM IX.

*The Impetus, and the Angle of Elevation, being given; to find at what Diſtance the Piece ought to be planted, to hit an Object, whoſe Diſtance above, or below, the Level of the Piece, is alſo given.*

### SOLUTION.

As the radius, is to the ſine of twice the given elevation, ſo is the impetus, to half the horizontal amplitude, at that elevation *(by Prop.* 1.)

And, as the radius, is to the co-tangent of the elevation, ſo is twice the perpendicular height, or depreſſion, of the object, to a fourth-proportional; which take from, or add to, half the horizontal amplitude, according as the object is elevated, or depreſſed; then find a mean proportion between the half-amplitude, and the ſum, or remainder; which, added to the ſaid half-amplitude, gives the diſtance ſought *(by Prop.* 2.)

Ex. Let the impetus be 3000 feet, the elevation 40°, and the height of the object 200 feet.

Then,

Digitized by Google

| | |
|---|---|
| Then, as radius . . . | 10. 0000 |
| to ſine . 80° . . . . | 9. 9933 |
| ſo is . 3000 . . . . | 3. 4771 |
| to . . 2954 . . . . | 13. 4704 |
| And, as radius . . . . | 10. 0000 |
| is to co-t. 40° . . . | 10. 0762 |
| ſo is . 400 . . . . | 2. 6021 |
| to . . 477 . . . . | 12. 6783 |

Now, the difference between the two numbers above found is 2477; whoſe log. being added to that of 2954, and the ſum divided by 2, the quotient will be the log. of 2705, the required mean proportional: whence the diſtance fought comes out 5659 feet.

## III. *Of Projections on Planes, inclined to that of the Horizon.*

### PROBLEM X.

*The Inclination of the Plane, and the Elevation and Impetus of the Piece, being known; to find the Amplitude of the Projection.*

#### SOLUTION.

As the ſquare of the co-ſine of the plane's inclination to the horizon, is to the rectangle of the ſine of the elevation above the plane and the co-ſine of the elevation above the horizon, ſo is 4 times the impetus, to the amplitude of the projection *(by Prop.* 3. *Cor.* 3.)

Ex. 1. Let the impetus be 4000 feet, the elevation 32° 30′, and the aſcent of the plane 8° 15′. Then the elevation above the plane will be 24° 15′; and the operation as follows.

8°

Digitized by Google

| | | |
|---|---|---|
| 8° 15′, co-arith. of its co-ſ. . | 0. 0045 | to be added |
| *The ſame repeated* . . . | 0. 0045 | |
| 24° 15′ . its ſine . . . | 9. 6135 | |
| 32° 30′ . its co-ſ. . . . | 9. 9260 | |
| 16000 f. its log. . . . | 4. 2041 | |
| 5658 feet the amp. req. . . | 2\|3. 7526 | |

Ex. 2. Let the elevation and impetus be the ſame as in the laſt example; but let the plane in this caſe have a deſcent of 8° 15′ (inſtead of an equal aſcent.)

Then the operation will ſtand thus,

| | | |
|---|---|---|
| 8° 25′, co-ar. of its co-ſ. . . | 0. 0045 | to be added |
| *The ſame repeated* . . . | 0. 0045 | |
| 40° 45′, its ſine . . . | 9. 8147 | |
| 32° 20′, its co-ſine . . . | 9. 9260 | |
| 16000 f. its log. . . . | 4. 2041 | |
| 8992 f. amp. req. . . . | 2\|3. 9538 | |

## PROBLEM XI.

*The Inclination of the Plane, the Elevation of the Piece, and the Amplitude of the Projection, being given; to find the Impetus.*

### SOLUTION.

As the rectangle of the ſine of the elevation above the plane and the co-ſine of the elevation above the horizon, is to the ſquare of the co-ſine of the plane's inclination, ſo is the given amplitude, to 4 times the required impetus *(by Prop. 3. Cor. 3.)*

Ex. Suppoſe the plane to have an aſcent of 8° 15′, and that the amplitude thereon, at an elevation of 32° 30′, is 5658 feet. Then, the elevation above the plane being 24° 15′, we ſhall have

24°

Digitized by Google

| | | |
|---|---|---|
| 24° 15′ co-ar. of its ſine . . | 0. 3865 | to be added |
| 32° 30′ co-ar. of its co-ſ. . . | 0. 0740 | |
| 8° 15′ its co-ſ. . . . | 9. 9955 | |
| *The ſame* . . . . | 9. 9955 | |
| 5658 f. its log. . . . . | 3. 7526 | |
| 16000 f. . . . . | 24. 2041 | |

¼ of which, or 4000 feet, is the impetus req.

## PROBLEM XII.

*The Inclination of the Plane, the Impetus, and the Amplitude being given, to determine the Elevation.*

### SOLUTION.

As the radius is to the co-ſine of the plane's inclination, ſo is the given diſtance, on the plane, to the horizontal diſtance correſponding: from which, the impetus, and the plane's (or object's) elevation or depreſſion, the elevation of the piece may be found, *by Prob.* 8.

Ex. Let the plane have an aſcent of 8° 15′, and let the given amplitude thereon be 5658 feet, ſuppoſing the *impetus* to be 4000 feet.

| | |
|---|---|
| Then, as the radius . . | 10. 0000 |
| is to the co-ſine of 8° 15′ | 9. 9955 |
| ſo is 5658 feet . . | 3. 7526 |
| to . 5600 feet . . | 13. 7481 |

the horizontal diſtance of the place where the ball impinges: as for the reſt of the operation, it is exactly the ſame as in the example to *Prob.* 8; for which reaſon it will be needleſs to repeat it here.



Digitized by Google

## PROBLEM XIII.

*Having the Angle of Elevation, or Depreſſion of the Object, together with the Elevation, and the Impetus of the Piece; to determine the Time of the Flight.*

### SOLUTION.

As the co-ſine of the elevation, or depreſſion of the object, is to the ſine of the elevation of the piece above the object, ſo is half the ſquare root of the number of feet in the *impetus* given, to the required number of ſeconds in the flight.

Ex. Suppoſe the elevation of the object to be 8° 30′; that of the piece 45°; and the *impetus* 3600 feet.

Then, the ſquare root of 3600 being 60, we have

| | | |
|---|---|---|
| As the co-ſine 8° 30′ | co-ar. | 0. 0048 |
| is to the ſine 36° 30′ | . . | 9. 7744 |
| ſo is 30 | . . . | 1. 4771 |
| to 18,04 ſeconds req. | . . | 1\|1. 2563 |

## IV. *Of the* MAXIMA *and* MINIMA, *in the Motion of Projectiles.*

## PROBLEM XIV.

*The greateſt horizontal Amplitude being given, to find the greateſt Amplitude on a Plane whoſe Inclination to the Horizon is alſo given.*

### SOLUTION.

Take half the angle included between the plane and the zenith; the complement of which is the required elevation *(by Prop.* 3. *Cor.* 6.) Then ſay, as the tangent of the ſaid elevation, is to the tangent of the plane's inclination, ſo is the given amplitude, to the difference between it and the amplitude ſought *(by Prop.* 3. *Corol.* 7.)

Ex.

Digitized by Google

Ex. Let the greateſt horizontal range be 8000 feet, and the inclination of the plane 12° 30′, deſcending.

Here the angle included between the plane and the zenith, or vertical point, being 102° 30′, the half thereof will be 51° 15′, and the elevation 38° 45′.

| | | |
|---|---|---|
| Theref. as tang. 38° 45′ | . | 9. 9045 |
| is to tang. . 12° 30′ | . | 9. 3457 |
| ſo is . . . 8000 f. | . | 3. 9031 |
| to . . . . 2210 f. | . | 3. 3443 |

Which, added to 8000 (becauſe the plane deſcends) gives 10210 feet, for the true anſwer in this caſe.

## PROBLEM XV.

*The greateſt Amplitude, on an inclined Plane, being given, to find the greateſt Amplitude, on the Plane of the Horizon.*

### SOLUTION.

As the radius, is to the ſine of the plane's inclination, ſo is the given amplitude, to the difference between *it* and the required amplitude *(by Prop.* 3. *Corol.* 8.)

Ex. Let the inclination of the plane be 12° 30′, deſcending, and the given amplitude 10210 feet.

| | | |
|---|---|---|
| Then, as radius . | . . | 10. 0000 |
| is to *S.* 12° 30′ . | . . | 9. 3353 |
| ſo is . 10210 . | . . | 4. 0090 |
| to . . 2210 . | . . | 3. 3443 |

Which taken from 10210, gives 8000 feet, for the true diſtance ſought.



Digitized by Google

## PROBLEM XVI.

*The Impetus, and the perpendicular Height, or Depreſſion, of an Object being given, to find the greateſt horizontal Diſtance at which the Ball can, poſſibly, hit the Object, and alſo the Elevation anſwering thereto.*

### SOLUTION.

Take the difference of the two given quantities, if the object be elevated; but otherwiſe, their ſum; then find a mean proportional between the Impetus and ſaid difference, or ſum; the double of which will be the diſtance ſought *(by Prop. 3. Cor. 9.)*

For the elevation, it will be, as the diſtance thus found is to the height, or depreſſion of the object, ſo is the radius to the tangent of an angle; which added to, or ſubtracted from, 90 degrees, reſpectively, gives the double of the required elevation *(by Prop. 3. Cor. 9.)*

Ex. Let the impetus be 4000 feet, and the depreſſion of the object, below the level of the piece, 2210 feet.

Here we are to take a mean propoportional between 4000 and 6210; which is $= \sqrt{4000 \times 6210} = 4984$; whoſe double 9968 it the required diſtance.

| | | |
|---|---|---|
| Moreover, we have, as | 9968 | 3. 9986 |
| is to | 2210 | 3. 3443 |
| ſo is radius | | 10. 0000 |
| to tang. | 12° 30′ | 9. 3457 |

whence the elevation appears to be 38° 45′

From this problem, the greateſt amplitude of a ball, projected from a given height above the level of the horizon, is given.

PROBLEM

Digitized by Google

## PROBLEM XVII.

*Having the horizontal Diſtance of the Object, together with its perpendicular Height, or Depreſſion, above or below the Level of the Piece, to determine the leaſt Impetus, poſſible, whereby the Ball may reach the Object; and alſo the Elevation correſponding.*

### SOLUTION.

As the horizontal diſtance of the object is to its perpendicular height or depreſſion, ſo is the radius to the tangent of an angle; which added to, or ſubtracted from 90 degrees, gives the double of the elevation *(by Prop. 3. Cor. 6.)*

And, as the radius is to the tangent of the elevation, ſo is the given horizontal diſtance, to twice the impetus required *(by Prop. 3. Corol. 11.)*

Ex. Let the horizontal diſtance be 9968 feet, and the diſtance of the object below the level of the piece 2210 feet.

| | | |
|---|---|---|
| Then, as . | 9968 . . | 3. 9986 |
| is to . . | 2210 . . | 3. 3443 |
| ſo is rad. . | . . . | 10. 0000 |
| to tang. | 12° 30′ . . | 9. 3457 |

Therefore the elevation is 38° 45′; and it will be

| | | |
|---|---|---|
| As radius | . . . | 10. 0000 |
| is to tang. | 38° 45′ . . | 9. 9945 |
| ſo is . | 9968 . . | 3. 9986 |
| to . . | 8000 . . | 3. 9031 |

The half of which, or 4000 feet, is the impetus ſought.

PROBLEM

## PROBLEM XVIII.

*To find the greateſt Height a Ball can, poſſibly, reach in a Perpendicular to the Horizon; the Impetus and the Diſtance of the Piece from the ſaid Perpendicular, being given.*

### SOLUTION.

Find a third proportional to the impetus and half the given diſtance, which ſubtract from the impetus, and the remainder will be the anſwer *(by Prop. 3. Corol.* 10.)

Thus, if the impetus be 4000 f. and the given diſtance 6000 feet, then it will be, as 4000 : 3000 : : 3000 : 2250; which taken from 4000, leaves 1750 feet, for the greateſt height the ball can poſſibly reach in the propoſed perpendicular, with that impetus.

PART

Digitized by Google

# PART IV.

## EXHIBITING

*A new, and very comprehenſive Method for extracting the Roots of Equations in Numbers; by increaſing the Dimenſions of the unknown Quantity.*

IN this method (as in the common method by converging ſeries) the value ſought is, firſt of all, to be nearly eſtimated (either from the equation itſelf, or from the nature of the problem whence it is derived); and ſome unknown quantity (as $z$) muſt be aſſumed to expreſs the difference between that value (which we will denote by $r$) and the true value required: and then, by ſubſtituting $r \pm z$ inſtead of its equal, in the given equation, a new equation will emerge, affected only with $z$ and known quantities.

The equation being thus prepared for a ſolution, multiply it into two, or more ſucceeding terms of the ſeries $1+Az+Bz^2+Cz^3+Dz^4$ &c. (according to the degree of exactneſs neceſſary) and let the coefficients of the homologous terms of the new equation hence ariſing, wherein the ſquare, and the next, higher powers of $z$ are concerned, be made equal to nothing: by which means the value of A, &c. will be determined, and as many terms of the equation deſtroyed, at the ſame time, as there are terms in the aſſumed multiplicator *minus* one. And the terms involving the ſuperior powers, which yet remain, being of very ſmall value, may alſo be rejected: whence the equation will be reduced to a ſimple one: from which the value of $z$ will be found.

Let

Digitized by Google

Ex. 1. Let the equation propounded be $x^2=50$.

Here, $x$ being ſomething greater than 7, put $7+z=x$; then the given equation will become $49+14z+z^2=50$;
or, $-1+14z+z^2=0$. This, multiplied by $1+Az+Bz^2$,

gives $\left\{\begin{array}{lll} -1+14z+z^2 & * & * \\ *-Az+14Az^2+Az^3 & * & \\ * & *-Bz^2+14Bz^3+Bz^4 & \end{array}\right\}=0.$

Whence, by equating the coefficents of the like terms, we have $1+14A-B=0$, and $A+14B=0$.

From which equations A is found $=-\frac{14}{197}$; and B $\left(=-\frac{A}{14}\right)=\frac{1}{197}$:
and, by ſubſtituting theſe values above,

we have $-1+14+\frac{14}{197}\times z+\frac{z^4}{197}=0.$

Which, by rejecting the exceeding ſmall quantity $\frac{z^4}{197}$,

becomes $-1+14+\frac{14}{197}\times z=0.$

Hence $-197+2772z=0$;

and $z=\frac{197}{2772}=.0710678$, nearly: which value is true to the laſt decimal place: and, if more terms of the ſeries $1+Az+Bz^2+Cz^3$ &c. had been taken, the concluſion would have been ſtill exacter in proportion.

Ex. 2. Suppoſe the given equation (when prepared for a ſolution) to be $-2+5z-z^3=0$.

Here, if four terms of the general ſeries $1+Az+Bz^2+Cz^3$ &c. be taken, and multiplied by $-2+5z-z^3$, our equation will be changed into the following one,

*viz.*

Digitized by Google

*viz.* $\left\{\begin{array}{llll} -2-2Az-2Bz^2-2Cz^3 & * \\ *+5z & +5Az^2+5Bz^3+5Cz^4 \\ * & * \quad * & -z^3-Az^4 \ \&c. \end{array}\right\}=0.$

Where, by comparing the homologous terms, we have $2B=5A$, $2C=5B-1$, and $5C=A$.

Hence, $25B-5$ $(=10C)$ $=2A$; but, by the firſt of theſe equations, $B=\frac{5A}{2}$; therefore $\frac{125A}{2}-5=2A$, and conſequently $A=\frac{10}{121}$.

Let this value be now wrote inſtead of A; by which means our laſt equation for the value of $z$ (neglecting the terms $-Bz^5-Cz^6$) is reduced to $-2+5-\frac{20}{121}\times z=0$.

Whence $z$ comes out $=\frac{242}{585}=0.414$ nearly.

Let there be now given the general equation.

$$-p+az+bz^2+cz^3+dz^4 \ \&c. =0.$$

Then, multiplying by $(1+Az)$ the two firſt terms of the ſeries, only,

we get $\left\{\begin{array}{ll} -p+az+bz^2 \ \&c. \\ *-pAz+aAz^2 \ \&c. \end{array}\right\}=0.$

Here, by making $b+aA=0$, A is found $=-\frac{b}{a}$; and our equation becomes $-p+a+\frac{bp}{a}\times z=0$: whence $z$ is given $=\frac{p}{a+\frac{bp}{a}}=\frac{ap}{aa+bp}$.

The value of $z$, here determined, taking in two terms of the given ſeries, $az+bz^2+cz^3$ &c. I call an approximation of the ſecond degree (as the common method of converging ſeries, which takes in the firſt



Digitized by Google

term only, may, by the ſame rule, be called an approximation of the firſt degree.)

But to obtain an approximation of the third degree, or ſuch an one as ſhall include three terms of the original ſeries, let the given equation, $-p+az+bz^2+cz^3$ &c. $=0$, be now multiplied by three terms of the aſſumed ſeries $1+Ax+Bz^2$ &c: whence there ariſes this

new equation $\left\{\begin{array}{llll} -p+az+bz^2+cz^3 & \&c. \\ *-Apz+Aaz^2+Abz^3 & \&c. \\ * \quad * \quad -Bpz^2+Baz^3 & \&c. \end{array}\right\}=0.$

Where, by equating the homologous terms, we have $b+Aa-Bp=0$, and $c+Ab+Ba=0$.

Let the former of theſe equations be multiplied by $a$, and the latter by $p$; and then add the two products together; ſo ſhall $ab+Aa^2+pc+Abp=0$; and conſequently $A=-\frac{ab+cp}{aa+bp}$. Whence $z\ \left(=\frac{p}{a-pA}\right)$ is likewiſe given.

To have an approximation of the fourth degree, four terms of the ſeries $1+Az+Bz^2$ &c. muſt be taken, for a multiplicator: by which means the equation given will be

transformed to $\left\{\begin{array}{llll} -p+az+bz^2+cz^3+dz^4. & \&c. \\ *-Apz+Aaz^2+Abz^3+Acz^4. & \&c. \\ * \quad * \quad -Bpz^2+Baz^3+Bbz^4. & \&c. \\ * \quad * \quad * \quad -Cpz^3+Caz^4. & \&c. \end{array}\right\}=0.$

Here we have

$$B=\frac{Aa}{p}+\frac{b}{p},$$

$$C=\frac{Ba}{p}+\frac{Ab}{p}+\frac{c}{p},$$

$$0=Ca+Bb+Ac+d,$$

from whence, exterminating C, there ariſes

$$\frac{Ba}{p}+\frac{Ab}{p}+\frac{c}{p}+\frac{Bb}{a}+\frac{Ac}{a}+\frac{d}{a}=0,$$

or

Digitized by Google

or $\overline{\frac{a}{p}+\frac{b}{a}} \times B + \overline{\frac{b}{p}+\frac{c}{a}} \times A + \frac{c}{p}+\frac{d}{a}=0.$

Which, by ſubſtituting the value of B,

becomes $\overline{\frac{a}{p}+\frac{b}{a}} \times \overline{\frac{Aa}{p}+\frac{b}{p}} + \overline{\frac{b}{p}+\frac{c}{a}} \times A + \frac{c}{p}+\frac{d}{a} = 0;$

that is, $\overline{\frac{aa}{pp}+\frac{2b}{p}+\frac{c}{a}} \times A + \frac{ab}{pp}+\frac{bb}{ap}+\frac{c}{p}+\frac{d}{a} = 0.$

Multiply the whole by $ap^2$; then will $\overline{a^3+2abp+cpp} \times A+a^2b+b^2p+acp+dp^2=0;$

whence A is found $= -\frac{aab+\overline{ac+bb} \times p+dpp}{a^3+2bp+cpp};$

and $z = \frac{p}{a-pA}$, *as in the preceding caſe.*

By the ſame method, an approximation of any higher degree, to take in as many terms of the propoſed ſeries as you pleaſe, may be derived.

For, it is evident, from above, that in all caſes whatever,

B will be $= \frac{Aa}{p}+\frac{b}{p},$

$C = \frac{Ba}{p}+\frac{Ab}{p}+\frac{c}{p},$

$D = \frac{Ca}{p}+\frac{Bb}{p}+\frac{Ac}{p}+\frac{d}{p},$

$E = \frac{Da}{p}+\frac{Cb}{p}+\frac{Bc}{p}+\frac{Ad}{p}+\frac{e}{p},$

&c. &c.

where the laſt value is, always, equal to nothing.



Digitized by Google

Therefore, by making

$$\begin{cases} Q = \frac{a}{p} \\ R = \frac{aQ+b}{p} \\ S = \frac{aR+bQ+c}{p} \\ T = \frac{aS+bR+cQ+d}{p} \\ V = \frac{aT+bS+cR+dQ+e}{p} \\ \&c. \end{cases}$$

and

$$\begin{cases} q = \frac{b}{p} \\ r = \frac{aq+c}{p} \\ s = \frac{ar+bq+d}{p} \\ t = \frac{as+br+cq+e}{p} \\ v = \frac{at+bs+cr+dq+f}{p} \\ \&c. \end{cases}$$

the value of $A$ will be determined, by dividing the laſt of theſe quantities $q, r, s, t$, &c. by the correſponding quantity of the upper ſeries Q, R, S, &c. and changing the ſign of the quotient.

Whence we get $\frac{p}{a+p\times\frac{q}{Q}}$, $\frac{p}{a+p\times\frac{r}{R}}$, $\frac{p}{a+p\times\frac{s}{S}}$, &c.

or, $\frac{1}{\frac{a}{p}+\frac{q}{Q}}$, $\frac{1}{\frac{a}{p}+\frac{r}{R}}$, $\frac{1}{\frac{a}{p}+\frac{s}{S}}$, &c. for ſo many different values of $z$; whereof each is more exact than the preceding one.

Theſe

Theſe equations are eaſily derived from thoſe above, expreſſing the relation of the quantities A, B, C, D, &c.

For, by writing Q and $q$, inſtead of their equals, in the firſt of thoſe equations $\left(B=\frac{Aa}{p}+\frac{b}{p}\right)$ it becomes $B=QA+q$.

Which value of B, wrote in the 2d equation, $C=\frac{Ba}{p}+\frac{Ab}{p}+\frac{c}{p}$, gives

$C=\frac{aQA}{p}+\frac{aq}{p}+\frac{Ab}{p}+\frac{c}{p}=RA+r$; by ſubſtituting R and $r$ in the room of their equals.

Moreover the third equation, by ſubſtituting for C and B, becomes $D=\frac{aRA}{p}+\frac{ra}{p}+\frac{bQA}{p}+\frac{bq}{p}+\frac{Ac}{p}+\frac{d}{p}=SA+s$.

After the very ſame manner $E=TA+t$, $F=VA+v$, &c. And, by putting all theſe ſeveral values, ſucceſſively, equal to nothing, we have $-\frac{q}{Q}$, $-\frac{r}{R}$, $-\frac{s}{S}$, &c. for ſo many different approximations of the value A. Which being ſubſtituted in the general equation $z=\frac{p}{a-pA}$, the very expreſſions before given, for the value of $z$, are obtained.

It will be proper, now, to ſhew the uſe of the ſeveral approximations, or theorems, derived above, by a few examples.

In

In the firſt place, then, let the equation given be $x^3 = 10$.

This equation, by making $2+z=x$, will be transformed to $-2+12z+6z^2+z^3=0$: which being compared with the general equation
$-p+az+bz^2+cz^3+dz^4$ &c. $=0$,
we have $p=2$, $a=12$, $b=6$, $c=1$, $d=0$, &c.

Therefore (by the firſt of the three approximations, at *p.* 217) $-A\ \left(=\frac{b}{a}\right)$ is found $=\frac{1}{2}$.

Whence $z\ \left(=\frac{p}{a-Ap}\right)$ comes out $=\frac{2}{13}=0.1538$, nearly.

But, according to the ſecond approximation, or theorem, the value of $-A\ \left(=\frac{ab+cp}{aa+bp}\right)$ will be $=\frac{72+2}{144+12}=\frac{37}{78}$: and conſequently that of $z\ \left(=\frac{p}{a-Ap}\right)=\frac{78}{505}=0.15445$.

Laſtly, the value of $-A$, according to the third approximation being $=\frac{a^2b+\overline{ac+bb}\times p+dp^2}{a^3+2abp+cpp}=\frac{144\times 6+96}{144\times 12+144\times 2+4}=\frac{36\times 6+24}{36\times 14+1}=\frac{48}{101}$, the correſponding value of $z$ will therefore come out $=\frac{101}{654}=0.154434$: which number is true in all its places.

The very ſame concluſions will likewiſe be brought out from the general ſolution.

For,

Digitized by Google

For, $p$ being here $= 2$, $a = 12$, $b = 6$, $c = 1$, $d = 0$, &c. *(as before.)*

we have $Q\left(=\frac{a}{p}\right) = 6$,

$$R\left(\frac{aQ+b}{p}\right) = \frac{12\times 6+6}{2} = 39,$$

$$S\left(\frac{aR+bQ+c}{p}\right) = \frac{505}{2},$$

&c.

Alſo, $q\left(\frac{b}{p}\right) = 3$,

$$r\left(\frac{aq+c}{p}\right) = \frac{37}{2},$$

$$s\left(\frac{ar+bq+d}{p}\right) = 120,$$

&c.

Therefore $\frac{q}{Q} = \frac{1}{2}$, $\frac{r}{R} = \frac{37}{78}$, $\frac{s}{S} = \frac{240}{505} = \frac{48}{101}$; and conſequently $\frac{1}{\frac{a}{p}+\frac{q}{Q}} = \frac{1}{6+\frac{1}{2}} = \frac{2}{13}$, &c. *as before.*

For a ſecond example, ſuppoſe $10z - z^3 = 2$, or $-2 + 10z - z^3 = 0$.

In this caſe $p = 2$, $a = 10$, $b = 0$, $c = -1$, $d = 0$, &c. And therefore, for an approximation of the fourth degree, we have $-A\left(\frac{aab + \overline{ac + bb} \times p + dpp}{a^3 + 2abp + cpp}\right) = \frac{-20}{1000-4} = \frac{-5}{249}$; and conſequently $z\left(=\frac{p}{a-pA}\right) = \frac{249}{1240} = 0.2008045$; which value is true to the laſt figure.

For

For a third example, let there be given the equation

$$x^3+36x^2+432x=2272.$$

Here the value of $x$ appears to be about 4; let therefore $4+z$ be wrote for $x$; whence our equation is reduced to

$$96+768z+48z^2+z^3=0.$$

Which being compared with

$$-p+az+bz^2+cz^3+dz^4 \text{ \&c.} =0,$$

we have, in this caſe,

$p=-96$, $a=768$, $b=48$, $c=1$, $d=0$, \&c.

or, $\frac{a}{p}=-8$, $\frac{b}{p}=-\frac{1}{2}$, $\frac{c}{p}=-\frac{1}{96}$, \&c.

Hence $Q\left(\frac{a}{p}\right)=-8$,

$$R\left(\frac{aQ}{p}+\frac{b}{p}\right)=64-\tfrac{1}{2}=\frac{127}{2},$$

$$S\left(\frac{aR}{p}+\frac{bQ}{p}+\frac{c}{p}\right)=-\frac{48385}{96};$$

\&c.

Alſo $q\left(\frac{b}{p}\right)=-\frac{1}{2}$,

$$r\left(\frac{aq}{p}+\frac{c}{p}\right)=4-\frac{1}{96}=\frac{383}{96},$$

$$s\left(\frac{ar}{p}+\frac{bq}{p}+\frac{d}{p}\right)=-\frac{380}{12},$$

\&c.

Therefore $\frac{q}{Q}=\frac{1}{16}$, $\frac{r}{R}=\frac{383}{127\times 48}=\frac{383}{6096}$, $\frac{s}{S}=\frac{380\times 8}{48385}=\frac{3040}{48385}$;

From whence $z=-\frac{16}{127}=-0.12598$, nearly;

or, $z=-\frac{6096}{48385}=-0.1259894$, *more* nearly,

or, $z=-\frac{9677}{76808}=-0.125989480\,2$, *ſtill* nearer.

Laſtly,

Digitized by Google

Laſtly, let there be given the equation

$$\frac{x^2}{2}-\frac{x^4}{2.3.4}+\frac{x^6}{2.3.4.5.6}-\frac{x^8}{2.3.4.5.6.7.8}\ \&c. = \frac{1}{2}.$$

Then, making $z=x^2$, &c. we have

$$-1+z-\frac{z^2}{12}+\frac{z^3}{12\times 30}-\frac{z^4}{12\times 30\times 56}\ \&c. = 0.$$

Here $p=1$, $a=1$, $b=-\frac{1}{12}$, $c=\frac{1}{12\times 30}$, $d=-\frac{1}{12\times 30\times 56}$: &c.

And, by ſubſtituting theſe values in the third general approximation *(vid. p.* 219), we have $-A\left(=\frac{aab+\overline{ac+bb}\times p+dpp}{aaa+2abp+cpp}\right)$

$$=\frac{-\frac{1}{12}+\frac{1}{12\times 30}+\frac{1}{12\times 12}-\frac{1}{12\times 30\times 56}}{1-\frac{2}{12}+\frac{1}{12\times 30}}=-\frac{30\times 56-56-140+1}{10\times 30\times 56+56}=-\frac{1485}{16856}.$$

Therefore $z\left(=\frac{p}{a-pA}\right)=\frac{16856}{15371}=1.09661$;

and $x\ (=\sqrt{z})\ =1.04719$.

After the ſame manner the roots of other equations may be approximated: but I ſhall here ſhew, how the general theorems themſelves may be rendered more commodious, for certain particular caſes, by means of a proper transformation.

It is known, if two quantities be, reſpectively, increaſed, or decreaſed by two other, ſmall, quantities, nearly in the ſame ratio with the two firſt, that the ſums, or differences will ſtill be in the ſame ratio with the two firſt quantities, very near.



Digitized by Google

Wherefore, ſeeing the numerator of the fraction $\frac{ab+cp}{aa+bp}$, expreſſing the 2$^d$. general value of $-A$, *(vid. p.* 218) is in proportion to the denominator, nearly, as $b$ to $a$, or $cp$ to $\frac{acp}{b}$; let $cp$ be therefore taken from the numerator, and $\frac{acp}{b}$ from the denominator; agreeable to the above obſervation: by which means the fraction itſelf will be transformed to $\frac{ab}{aa+bp-\frac{acp}{b}} = \frac{b}{a+\overline{\frac{b}{a}-\frac{c}{b}}\times p} = \frac{b}{a+rp}$; ſuppoſing $r=\frac{b}{a}-\frac{c}{b}$.

And, in the very ſame manner the fraction $\frac{aab+\overline{ac+bb}\times p+dpp}{aaa+2abp+cpp}$, expreſſing the third value of $-A$, is changed to $\frac{aab+\overline{ac+bb}\times p}{aaa+2abp+cpp-\frac{adpp}{b}} = \frac{aab+\overline{ac+bb}\times p}{a^3+2abp+sp^2}$ (by putting $s=c-\frac{ad}{b}$).

But this laſt value is ſtill capable of a further reduction: for, the ratio of the numerator to the denominator being that of $\frac{bsp}{r}$ to $\frac{asp}{r}+sp^2$, nearly (as appears from the preceding caſe) let, therefore, the former of theſe quantities be ſubtracted from the numerator, and the latter from the denominator: whence the fraction itſelf becomes

$$\frac{a^2b+\overline{ac+bb}\times p-\frac{bsp}{r}}{a^3+2abp-\frac{asp}{r}} = \frac{ab+c+\frac{bb}{a}-\frac{bs}{ra}\times p}{a^2+2b-\frac{s}{r}\times p}$$

=

$= \frac{ab + \overline{c + bw} \times p}{aa + b + aw \times p}$; by putting $w = \frac{b}{a} - \frac{s}{ar}$ $(= \frac{b}{a} - \frac{cb - ad}{bb - ac})$.

Now, to exemplify the uſe of the theorems thus transformed, let the equation $96 + 768z + 48z^2 + z^3 = 0$ (given at p. 224) be here reſumed:

Then, in this caſe, $p$ being $= -96$, $a = 768$, $b = 48$, $c = 1$, $d = 0$, &c. we have $r$ $(= \frac{b}{a} - \frac{c}{b}) = \frac{1}{24}$, $s$ $(= c - \frac{ad}{b}) = 1$, and $w$ $(= \frac{b}{a} - \frac{s}{ar}) = \frac{1}{32}$.

Whence, according to the firſt approximation,

$-A$ $(= \frac{b}{a + rp}) = \frac{48}{768 - 4} = \frac{12}{191}$.

And, according to the ſecond,

$$-A \left(= \frac{ab + \overline{c + bw} \times p}{aa + \overline{b + aw} \times p}\right) = \frac{768 \times 48 + \overline{1 + \frac{1}{2}} \times -96}{768 \times 768 + \overline{48 + 24} \times -96}$$

$$= \frac{768 - 5}{768 \times 16 - 72 \times 2} \text{ (by dividing every term by 48)}$$

$$= \frac{763}{12144}.$$

Hence the value of $z$ $(= \frac{1}{\frac{a}{p} - A})$ comes out $= -\frac{191}{1516} = -0,1259894$, *nearly*; or equal to $-\frac{12144}{96389} = -0.125989480\overline{2}$, *more nearly*.

Theſe concluſions agree with thoſe before given, by the former method. But the laſt general approximations, containing the feweſt dimenſions of the quantity $p$, will commonly be found to have the advantage, in point of expedition, when the value of that quantity conſiſts of ſeveral decimal



Digitized by Google

places, and alſo when the coefficients $a$, $b$, $c$, $d$, of the powers of the unknown quantity $z$, are related to each other according to ſome unknown law.

Of this kind are the coefficients of ſuch ſeries as ariſe in extracting the roots of pure powers; and in theſe caſes the general theorems, or equations themſelves, are capable of being rendered ſtill more commodious.

Let there be aſſumed the equation $x^n = k$ (which includes all the caſes of pure, or ſimple powers, according to the value of the exponent $n$).

Then, by aſſuming $r$ nearly equal to $x$, and making $r \times \overline{1+z} = x$, we ſhall have $r^n \times \overline{1+z}^n = k$:

and therefore $-\frac{k}{r^n} + \overline{1+z}^n = 0$; that is

$$-\frac{k}{r^n} + 1 + nz + \frac{n}{1} \times \frac{n-1}{2} z^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{2} z^3$$

$\&c. = 0$; or laſtly $-p + z + \frac{n-1}{2} z^2 + \frac{n-1}{2} \times \frac{n-2}{3} z^3 + \&c. = 0$; by dividing the whole by $n$, and putting $p = \frac{k - r^n}{nr^n}$.

Here (by a compariſon with the general equation $-p + az + bz^2 \&c. = 0$) we have $a = 1$, $b = \frac{n-1}{2}$, $c = \frac{n-1}{2} \times \frac{n-2}{3}$, $d = \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, $\&c.$

Therf. $r \left(\frac{b}{a} - \frac{c}{b}\right) = \frac{n-1}{2} - \frac{n-2}{3} = \frac{n+1}{6}$,

$$s \left(c - \frac{ad}{b}\right) = \frac{n-2}{3} \times \overline{\frac{n-1}{2} - \frac{n-3}{4}} = \frac{n-2}{3} \times \frac{n+1}{4},$$

$$w \left(\frac{b}{a} - \frac{s}{ar}\right) = \frac{n-1}{2} - \frac{n-2}{2} = \frac{1}{2},$$

Whence,

Digitized by Google

Whence, for an approximation of the third degree,

$$-A\left(=\frac{b}{a+rp}\right)=\frac{\frac{1}{2}\times\overline{n-1}}{1+\overline{n+1}\times\frac{1}{6}p};$$

and $z\left(=\frac{p}{a-Ap}\right)=\frac{p+\overline{n+1}\times\frac{1}{6}p^2}{1+\overline{2n-1}\times\frac{1}{3}p}$.

But, for an approximation of the fourth degree,

$$-A\left(=\frac{ab+\overline{c+bw}\times p}{aa+\overline{b+aw}\times p}\right)=\frac{\frac{1}{2}\times\overline{n-1}+\overline{n-1}\times\overline{2n-1}\times\frac{1}{12}p}{1+\frac{1}{2}np};$$

and $z=\dfrac{p+\frac{1}{2}npp}{1+\frac{2n-1}{2}\times p+\frac{\overline{2n-1}\times\overline{n-1}}{12}\times pp}$.

Hence it is manifeſt that the root $x$, of the propoſed equation $x^n=k$, is equal to

$$r+\frac{rp\times\overline{1+n-1}\times\frac{1}{6}p}{1+\overline{2n-1}\times\frac{1}{3}p},\ \textit{nearly};$$

or, equal to, $r+\dfrac{rp\times\overline{1+\frac{1}{2}np}}{1+\frac{2n-1}{2}\times p+\frac{\overline{2n-1}\times\overline{n-1}}{12}\times p^2}$,

*more nearly.*

But both theſe expreſſions, in caſes where $p$ is a proper fraction, will be better adapted to practice by making $\frac{nr^n}{k-r^n}=v\ \left(=\frac{1}{p}\right)$, and ſubſtituting $\frac{1}{v}$ for $p$, its equal: whence (after proper reduction)

$x=r+\frac{r}{v}\times\frac{6v+n+1}{6v+4n-2}$, *nearly*;

or, $x=r+\dfrac{r\times\overline{2v+n}}{v\times\overline{2v+2n-1}+\frac{1}{6}\times\overline{2n-1}\times\overline{n-1}}$, *more nearly.*

To

To ſhew now the uſe and great exactneſs of theſe laſt approximations, by an example, let it be propoſed to extract the ſquare root of 441. Here the general equation $x^n = k$, becoming $x^2 = 441$, we have $n = 2$, $k = 441$, and $v \left(= \frac{nr^n}{k - r^n}\right) = \frac{800}{41}$; ſuppoſing $r$ to be aſſumed $= 20$.

Therefore, by the firſt approximation,

$$x = 20 + \frac{41}{40} \times \frac{1641}{1682} - 20 + \frac{67281}{67280} = .21 + \frac{1}{67280},$$

nearly.

And, by the ſecond, $x = 20 + \frac{2758480}{2758481} = 21 - \frac{1}{2758481}$, more nearly.

Again, let there be given the equation $x^3 = 500$: then, aſſuming $r = 8$, we have $v \left(\frac{nr^n}{k - r^n}\right) = \frac{3 \times 512}{-12} = -128$.

Hence, by the firſt approximation,

$$x = 8 - \frac{1}{16} \times \frac{-768 + 4}{-768 + 10} = 8 - \frac{191}{3032} = 7{,}93700527,$$

nearly.

And, by the ſecond, $x = 8 - \frac{8 \times 253}{128 \times 251 + \frac{5}{3}} = 8 - \frac{6072}{96389} = 7.9370052599$, more nearly.

All the different approximations hitherto delivered were, originally, derived by multiplying the given equation into a certain number of terms of the aſſumed ſeries $1 + Az + Bz^2 + Cz^3$ &c. But there are other methods (though, perhaps, none ſo general) by which the ſecond, third, &c. dimenſions of the unknown quantity may, in like ſort, be deſtroyed (without aſ-

ſuming

Digitized by Google

ſuming any ſeries) and from thence the value of that quantity approximated, to what degree of exactneſs you pleaſe.

Let there, for inſtance, be given the equation,
$2z+z^2=1$, or $z^2=1-2z$.

Then, by ſquaring both ſides thereof, there ariſes $z^4=1-4z+4z^2$,

And if, inſtead of the laſt term $4z^2$, its equal, $4\times 1-2z$, be ſubſtituted, you will have $z^4=5-12z$: this, ſquared, gives $z^8=25-120z+144z^2=169-408z$, by ſubſtituting for $z^8$, as before.

Here, rejecting $z^8$ (on account of its ſmallneſs in compariſon of the other terms) we have $408z=169$, and therefore $z=\frac{169}{408}=0,414213$, nearly; which is true to the laſt figure, incluſive. But, if you would have the anſwer ſtill nearer the truth, let the above equation $z^8=169-408z$ be, either, multiplied, again, by itſelf, or into ſome one of the preceding ones. Thus, if it be multiplied into $z^4=5-12z$, you will have
$z^{12}=845-4068z+4896z^2=5741-13860z$:
where, $z^{12}$ being rejected, $z$ is found $=\frac{5741}{13860}=0.41421356$, *&c.*

Again, if there be given the equation $z^3=3z-p$; then, by ſquaring both ſides thereof, we have
$z^6=9z^2-6pz+p^2$:
and therefore $z^7=9z^3-6pz^2+p^2z=-6pz^2+\overline{pp+27}\times z-9p$; by writing $27z-9p$ inſtead of its equal $9z^3$.

Now, to the triple of this laſt equation, let the 2d. equation, multiplied by $2p$, be added:

By

Digitized by Google

By which means $z^2$ will be exterminated, and you then will have $3z^7 + 2pz^6 = \overline{81 - 9pp} \times z - 27p + 2p^3$. Whence (rejecting $3z^7 + 2pz^6$) the value of $z$ is found $= \frac{\overline{27 - 2pp} \times p}{\overline{9 - pp} \times 9}$, nearly.

Various other expedients might be uſed, to exterminate the $2^d$, $3^d$, *&c.* powers of the unknown quantity; but what is already delivered may ſuffice.

PART

Digitized by Google

# PART V.

## GIVING

*Some Account of the Nature of* FLUXIONS; *together with the Inveſtigation of the fundamental Rules.*

1. IN the Doctrine of Fluxions all kinds of magnitudes are conſidered as generated by the continual motion of ſome of their bounds or extremes; as a line by the motion of a point; a ſurface by the motion of a line; and a ſolid by the motion of a ſurface. So likewiſe time may be conſidered as repreſented by a line, increaſing uniformly by the motion of a point: and, as quantities of all kinds whatever are capable of increaſe and decreaſe, they may be repreſented, in like manner, by lines, ſurfaces, or ſolids, conceived to be generated by motion.

2. *Every quantity thus generated is called a fluent, or flowing quantity: and the magnitude by which any flowing quantity would be uniformly increaſed, in a given time, with the generating celerity at any propoſed poſition, or inſtant* * *(was it from thence to continue invariable) is the fluxion of the ſaid quantity at that poſition, or inſtant.*

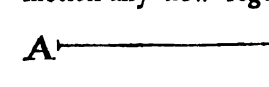
3. Thus,

* . *Some Authors define the generating celerity itſelf (and not the magnitnde it would produce) to be the fluxion; but uſe that magnitude as the meaſure of the ſaid celerity or fluxion; which is, in effect, coming to the ſame thing.*



Digitized by Google

3. Thus, let a point *m* be conceived to move from A, and thereby generate a right-line A*m*, with a motion any how regulated; and ſuppoſe the celerity thereof, at any propoſed poſition R, to be




ſuch, *as would*, (was it to continue invariable from that poſition) be ſufficient to deſcribe, or paſs uniformly over the diſtance R*r*, in the given time allowed for the fluxion; then will the ſaid diſtance R*r* truly expreſs the required fluxion of the flowing line A*m*, in that poſition.

4. It appears from hence, that, when the generating motion is uniform, the fluxion, and the increment *actually* deſcribed in the given time, are one and the ſame thing: but, if the velocity continually increaſes or decreaſes, the fluxion muſt then be either leſs, or greater than the ſaid increment, or the ſpace *actually* deſcribed: ſince an increaſe of the velocity muſt neceſſarily cauſe an increaſe in the diſtance gone over, and *vice verſa*.

5. If

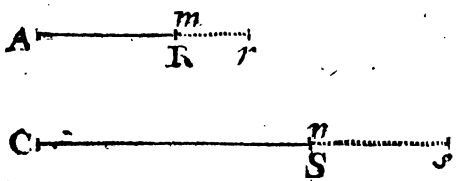
---

*Although, in forming a juſt and diſtinct conception of the nature, and quantity of a fluxion, the conſideration of time is, abſolutely, neceſſary (on which, even, our ideas of velocity depend), yet in the buſineſs and application of fluxions, it is not always requiſite, that ſome, vulgar (or common) meaſure of time (as a ſecond, minute, hour, &c.) ſhould be propounded for the production of the fluxions of the quantities under conſideration. A line generated by the uniform motion of a point, it is obſerved above, may be taken as a proper repreſentative, or meaſure of time: and that interval of time (be it what it will) wherein the line, ſo generated, is augmented by any length, or fluxion, aſſigned, may be taken as the time underſtood in the definition, allowed for the production of the fluxions of all other quantities that have any relation to, or dependence upon, the ſaid uniformly generated line. And theſe fluxions themſelves, by means of the ſaid relation and the given length, or fluxion, may be alſo truly exhibited, independent of any particular, known meaſure of time; as will hereafter, be fully made to appear.*

Digitized by Google

5. It appears moreover, from the above definition, that quantities, which flow, or increaſe together, ſo as to continue, *always*, in a conſtant ratio, have their fluxions, likewiſe, in *the ſame* conſtant ratio.

To illuſtrate this by a particular example, ſuppoſe two lines, A*m* and C*n*, to be ſo generated, by the uniform motion of two points *m* and *n*, that the latter of them ſhall be *always* equal to the double of the former: then, taking R, S, and *r*, *s* as cotemporary poſitions of the ſaid generating points, CS will be the double of AR, and C*s* the double of A*r*, by ſuppoſition; whence S*s*, the fluxion of CS, muſt of conſequence be the double of R*r*, the fluxion of AR.

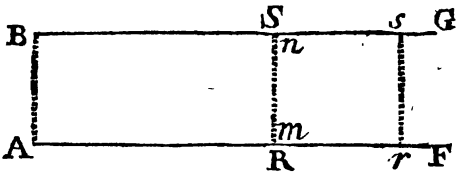



It is equally plain, on the other hand, that if the ratio of the fluents A*m*, C*n* is variable, that of the fluxions muſt alſo vary. Thus, if, while the point *m* continues to move uniformly on, at the rate of one inch (foot, yard, *&c.*) in a ſecond of time, the motion of the other Point *n* be ſo regulated that the number of inches (feet, yards, *&c.*) in the flowing line C*n* generated thereby, may be *always* equal to the ſquare of the number of thoſe in A*m* deſcribed by the former point *m*; then, in this caſe, it is manifeſt, that the ratio of the fluents A*m*, C*n* is a variable one; and that the ratio of the fluxions varies alſo; ſeeing the diſtances 1, 4, 9, 16, 25, *&c.* deſcribed in 1, 2, 3, 4, 5, *&c.* ſeconds of time, by the point *n*, increaſe much faſter in proportion than 1, 2, 3, 4, 5, *&c.* the correſponding diſtances gone over by the other point *m*, moving uniformly.



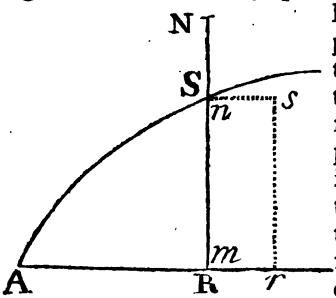
Digitized by Google

6. Hitherto regard has been had to the fluxions of lines: but the fluxions of ſuperfices and ſolids are conſidered in the ſame manner, and are comprehended with equal facility. Let a given right-line *mn*




be conceived to move parallel to itſelf, with an equable motion, from the poſition AB, and thereby generate the flowing rectangle A*mn*B; let alſo the diſtance R*r* be taken *(as above)* to expreſs the fluxion of the baſe A*m*, and let the rectangle R*rs*S be completed: then, this rectangle being the ſpace that is uniformly deſcribed by the generating Line *mn*, in the time that A*m* is in like manner increaſed by R*r*, it will therefore be the true fluxion of the flowing rectangle A*n*, by the definition. *Art.* 2.

7. The generation, and the fluxion of any triangular, or curvilineal, ſpace A S R, are conceived in like manner; by ſuppoſing a right-line *m*N to be carryed along, continually parallel to itſelf, ſo that the intercepted part thereof *mn* (which is itſelf a variable quantity) may paſs over, and thereby generate, the ſpace ARS propounded. And the fluxion of the ſpace thus generated, if R*r* be taken as the fluxion of the baſe (or abſciſſa) A*m*, will be truly expreſſed by the rectangle (R*s*) under R*r* and RS; as we ſhall have occaſion to ſhew more at large hereafter.




8. From.

Digitized by Google

8. From what has been thus far delivered, it will not be difficult to form a juſt idea of the fluxion of a ſolid: but it is time we now come to ſhew the manner of determining the fluxions of algebraic quantities; by means whereof all others, of what kind ſoever, are explicable: in order to which it will be requiſite, firſt of all, to premiſe the following obſervations.

1. *That, the final letters u, w, x, y, z, of the alphabet are uſually put for variable quantities; and the initial letters a, b, c, d, &c. for invariable ones:* thus, the *variable* baſe Am of the flowing rectangle Amn*B (in Art.* 6.) may be repreſented by $x$, and the *invariable* altitude *mn*, by $a$.

2. *That, the fluxion of a quantity repreſented by a ſingle letter is commonly expreſſed by the ſame letter with a dot, or full-point, over it:* thus the fluxion of $x$ is denoted by $\dot{x}$; and the fluxion of $y$ by $\dot{y}$.

3. *That, the fluxions of all quantities (having any relation to each other) are always to be taken, as contemporaneous, or ſuch as may be generated together, with their reſpective celerities, in one and the ſame time.*

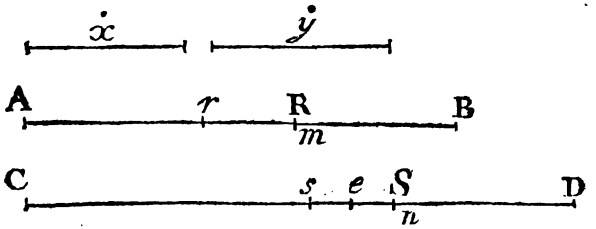
PROPOSITION

Digitized by Google

## PROPOSITION I.

9. *The Fluxion of any variable Algebraic Quantity being given, it is propoſed to find the Fluxion of the Square of that Quantity.*

Let two points *m* and *n* be ſuppoſed to move, at the ſame time, from two other, fixed, points A and C, along the right-lines AB and CD, in ſuch a manner, that the meaſure of the diſtance C*n*, deſcribed by the latter, may be, *always*, equal to the ſquare of the cotemporary diſtance A*m* deſcribed by the former point *m*, moving with an equable celerity.




Moreover, let R and S be any two contemporary poſitions of the ſaid points; and, ſuppoſing the deſcribed diſtances AR and CS to be denoted by $x$ and $y$, let the lines $\dot{x}$ and $\dot{y}$ be taken to repreſent the ſpaces that *would be* uniformly paſſed over, in the ſame given time, with the celerities of the ſaid points at R and S: ſo ſhall thoſe lines expreſs the fluxions of the variable quantities A*m* and C*n*, when the generating points *m* and *n* arrive at the foreſaid cotemporary poſitions R and S *(by the definition, Art.* 2).

Furthermore, if *r* and *s* be conſidered as any other correſponding places of the propoſed points, and the interval *r*R be denoted by $v$; then, AR being $=x$, and A*r* $=x-v$, we ſhall have CS $(=y)=x^2$, and C*s* $=\overline{x-v}^2$, *by hypotheſis*; and conſequently S*s* $(=$ CS $-$ C*s*$)$ $=2xv-vv$.

From

From which it appears, that, while the former point $m$ moves, uniformly, over the diſtance $v$, the other point $n$ paſſeth over a ſpace expreſſed by $2xv-vv$.

But this laſt diſtance, ſince the velocity of the generating point $n$ increaſes continually *(ſee Art.* 5.) is leſs than the ſpace that would be uniformly deſcribed, in the ſame time, with the velocity at S; and greater than that which would be deſcribed with the velocity at $s$; and, therefore, is equal to, and may be taken to expreſs, the ſpace which *might be* uniformly gone over by the celerity at ſome intermediate point $e$, between $s$ and S, in the ſame time.

Therefore, ſeeing the diſtance $(2xv-vv)$ that might be deſcribed with the celerity at the ſaid intermediate point $e$, is to the diſtance $(v)$ deſcribed by $m$, in the ſame time, as $2x-v$ to unity, it is evident that the ſaid (mean) celerity at $e$, muſt be to the celerity of $m$, in the ſame ratio; and conſequently, that, in the time the point $m$ would move over the given diſtance $\dot{x}$, the other point $n$, with its velocity at $e$, would deſcribe the diſtance $2x\dot{x}-v\dot{x}$: ſince the ſpaces deſcribed in equal times, by uniform motions are known to be as the velocities of the ſaid motions.

This

---

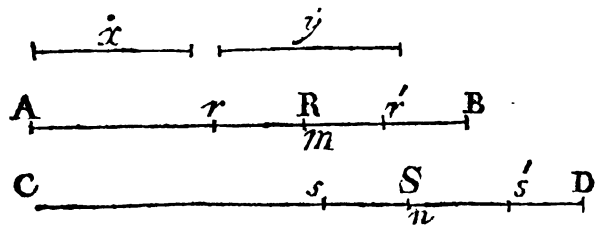
*The above method of inveſtigation hath been repreſented, as bearing a near affinity to the method of* Prime and Ultimate Ratios: *againſt which ſo many objections have been ſtarted, by the celebrated author of the Analyſt, as to embarraſs and ſtagger a great number of perſons; who are not well apprized how far the ſaid objections are juſtifiable, nor wherein their main force conſiſts. It is not my deſign to take a part in a diſpute, on which enough hath been already ſaid by others: though, that the method itſelf is perfectly ſcientific, I believe no one, that underſtands it, will deny; but whether the great inventor has therein expreſſed himſelf with all the caution and accuracy he was capable of, is another queſtion. Had he called that a* LIMITING-RATIO *which he names an* Ultimate-one, *the ingenious author above mentioned might not, perhaps, have found room for* his Ghoſts of departed Quantities. *However, be this as it will, theſe*

Digitized by Google

This being determined, let $r$ be now ſuppoſed to coincide with R, and $s$ with S, by means of the arrival of the generating points at R and S; then $e$, being always between $s$ and S, will likewiſe coincide with S; and the foreſaid diſtance $2x\dot{x}-v\dot{x}$, that might be uniformly deſcribed with the velocity at $e$ (now at S) will become, barely, $2x\dot{x}$; which *(by Art.* 2) is equal to $(\dot{y})$ the fluxion of Cn or $x^2$. From whence it appears, that the fluxion of the ſquare of any variable, or flowing quantity is found by multiplying twice the root, or quantity itſelf, into its fluxion.

*The ſame otherwiſe.*

10. Let the diſtance Ar be denoted by $u$, and let other things remain as before: then, CS being $=$




$xx$, and $Cs=uu$, *by ſuppoſition*, the diſtance $s$S, deſcribed in the ſame time with $r$R $(=x-u)$ will therefore be truly expreſſed by $xx-uu$, or its equal $\overline{x+u} \times \overline{x-u}$. Which diſtance, as the velocity of $n$ continually increaſes, muſt be greater than that which would be uniformly deſcribed, with the celerity at $s$, in the ſame time. Whence it is evident, that the velocity

*thoſe objections have nothing at all to do with our Method of inveſtigation given above. The preſſing (and only) difficulty, in the buſineſs of ultimate ratios, conſiſts, it is known, in conſidering the values of the quantities compared, in their ultimate ſtate, wherein their ratio is ſuppoſed to be taken: for*

Digitized by Google

velocity at $s$ is to the, uniform, velocity of $m$, in a leſs ratio than that of $\overline{x+u} \times \overline{x-u}$ to $x-u$, or of $\overline{x+u}$ to *unity*, or laſtly, of $\overline{x+u} \times \dot{x}$ to $\dot{x}$ (becauſe the velocities of uniform motions are as the ſpaces deſcribed in equal times.) Therefore, ſince the uniform velocity of $m$ (meaſured by the diſtance moved over in a given time) is defined by $\dot{x}$ it is plain that the meaſure of the velocity of the other point at $s$ (or the diſtance that *might be* uniformly deſcribed in the ſame given time) will be leſs than $\overline{x+u} \times \dot{x}$. Which quantity being, itſelf, leſs than $2x \times \dot{x}$ (becauſe $u$ is leſs than $x$) the celerity at $s$ muſt conſequently be leſs than $2x\dot{x}$, take the point $s$ where you will on this ſide of S.

Let now $r'$ and $s'$ be any other cotemporary poſitions of the two points, on the other ſide of R and S, and let A$r'$ be denoted by $w$: ſo ſhall the diſtance S$s'$, deſcribed in the ſame time with R$r'$ $(w-x)$, be truly expreſſed by $ww-xx$, or its equal $\overline{w+x} \times \overline{w-x}$ *(by hypotheſis.)* Which diſtance, as the velocity of the deſcribing

---

*for, if we look upon them as real magnitude, it is objected, that their ratio will not ſtrictly agree with the ultimate ratio aſſigned: and if, on the other hand, they be taken as mere nothings, we then loſe the very idea of proportion. But our inveſtigation, as is already obſerved, is not embarraſſed with any ſuch difficulty: for, though the diſtance* S s *grows, indeed, leſs and leſs, continually, and even vaniſhes when the generating point* n *arrives at* S; *yet the velocity of that point, which is the quantity in queſtion, neither vaniſhes, nor aſſumes a new law; but ſtill continues to increaſe in the ſame manner as before.—It may, poſſibly, be objected, that, as the meaſure of the ſaid velocity is, originally, derived by means of the diſtance* nS, *we cannot retain a clear idea of it, when that diſtance is vaniſhed out of the equation. But, with equal reaſon, it might be urged, that, we can have no juſt conception of the dimenſions and true proportion of a building, after the ſcaffolding by means of which it was raiſed, is taken away.*



Digitized by Google

cribing point $n$ continually increaſes, muſt, evidently, be leſs than that which would uniformly ariſe from the celerity at $s'$, in the ſame time. Whence it is alſo plain that the ſaid velocity at $s'$, is to the uniform velocity of $m$, in a greater ratio than that of $\overline{w+x \times w-x}$ to $\overline{w-x}$, or of $\overline{w+x \times \dot{x}}$ to $\dot{x}$. But the quantity $w+x \times \dot{x}$ is, itſelf, greater than $2x \times \dot{x}$, becauſe $w$ is greater than $x$; and ſo the meaſure of the ſaid velocity at $s'$ muſt conſequently be greater than $2x\dot{x}$.

Therefore, ſince the velocity increaſes continually, from C to D; and ſeeing the meaſure thereof, before the arrival of the generating point at S, is every where leſs, and afterwards every where greater, than $2x\dot{x}$; it is manifeſt, that, at S, it can be neither leſſer nor greater, but muſt have, or paſs through, the very value, or degree, expreſſed by $2x\dot{x}$. Q. E. I.

If the line A$m$ $(x)$ be ſuppoſed to be generated with an accelerated, or a retarded motion, inſtead of an uniform one, it will readily appear, from the firſt of the foregoing methods, that the required fluxion of $x^2$, ſuppoſing $\dot{x}$ to denote the meaſure of the velocity at R, will, *ſtill*, be expounded by $2x\dot{x}$.

For the ſpaces $rR$ $(v)$ and $sS$ $(2xv-vv)$ deſcribed in the ſame time, being to each other, in the ratio of $\dot{x}$ to $2x\dot{x}-v\dot{x}$, the mean celerities of the generating motions, at certain intermediate points between the extreme ones $r$, R, and $s$, S, muſt be likewiſe, in that ratio: which ratio, when $v$ becomes $=0$, and the points coincide, will become that of $\dot{x}$ to $2x\dot{x}$.

PROPOSITION

Digitized by Google

## PROPOSITION II.

11. *The Fluxions, $\dot{x}$ and $\dot{y}$, of two flowing Quantities, $x$ and $y$, being given; it is propoſed to determine the Fluxion of the Rectangle, or Product, $xy$, of the ſaid Quantities.*

Let $z$ be, *always*, equal to the ſum of the two propoſed quantities $x$ and $y$: then, the fluxions of equal quantities being alſo equal, $\dot{z}$ muſt be $=\dot{x}+\dot{y}$. Moreover, ſince $z$ is $=x+y$, we ſhall have $zz = xx + 2xy + yy$; and therefore $xy = \frac{1}{2}zz - \frac{1}{2}xx - \frac{1}{2}yy$. But the fluxion of $\frac{1}{2}zz - \frac{1}{2}xx - \frac{1}{2}yy$ (and conſequently that of its equal $xy$) appears, from the preceding propoſition, to be $z\dot{z} - x\dot{x} - y\dot{y}$: which, by writing $x+y$, and $\dot{x}+\dot{y}$, in the room of their equals $z$ and $\dot{z}$, will become $\overline{(x+y} \times \overline{\dot{x}+\dot{y}} - x\dot{x} - y\dot{y}) = y\dot{x} + x\dot{y}$, the required fluxion of $xy$. Hence it is apparent that the fluxion of the product, or rectangle, of any two flowing quantities, is expreſſed by the ſum of the products ariſing from the multiplication of each quantity into the fluxion of the other.

12. From the fluxion of a rectangle, above determined, the fluxion of a fraction, $\frac{z}{y}$, is very eaſily deduced.

For, by putting $x = \frac{z}{y}$ (the propoſed fraction) and multiplying both ſides of the equation by $y$ we have $xy = z$; and therefore $x\dot{y} + y\dot{x} = \dot{z}$, *as above*, (the fluxions of equal quantities being, neceſſarily, equal.) From this equation, by tranſpoſing $x\dot{y}$, and dividing the whole by $y$, we get $\dot{x} = \frac{\dot{z}}{y} - \frac{x\dot{y}}{y}$: this, by writing $\frac{z}{y}$ in the room of its equal $x$, becomes $\dot{x} = \frac{\dot{z}}{y} -$



Digitized by Google

$\frac{z\dot{y}}{yy} = \frac{y\dot{z} - z\dot{y}}{yy}$: which value is therefore the true fluxion of $x$, or, its equal, $\frac{z}{y}$, the fraction propoſed.

13. Moreover, from the fluxion of a rectangle, the fluxion of the continual product of three, four, five, or any other number, of flowing quantities, may be determined.

Thus, let the fluxion of $yzu$, where the number of factors is 3, be firſt required: then, by putting $x = zu$, our given expreſſion will be reduced to $yx$; and its fluxion will be $y\dot{x} + x\dot{y}$ *(by Prop.* 2.) But, $x$ being $= zv$, and therefore $\dot{x} = z\dot{v} + v\dot{z}$ *(by the ſame)*, if theſe values be ſubſtituted in $y\dot{x} + x\dot{y}$, it will become $y \times z\dot{v} + v\dot{z} + zv\dot{y} = yz\dot{v} + y\dot{z}v + \dot{y}zv$, the true fluxion of $yzu$, required.

Again, if the fluxion of $yzvw$, where the number of factors is four, was to be demanded; then, by making $x = zvw$, the quantity propoſed will be reduced to $yx$; and its fluxion will therefore be expreſſed by $y\dot{x} + x\dot{y}$; which, becauſe $x$ is $= zvw$, and $\dot{x} = zv\dot{w} + z\dot{v}w + \dot{z}vw$ *(as appears from above)* will be likewiſe expreſſed by $y \times zv\dot{w} + z\dot{v}w + \dot{z}vw + zvw\dot{y}$, or $yzv\dot{w} + yz\dot{v}w + y\dot{z}vw + \dot{y}zvw$.

In the ſame manner the fluxion of $yzvws$, will appear to be $yzv\dot{w}s + yzvw\dot{s} + yz\dot{v}ws + y\dot{z}vws + \dot{y}zvws$: and ſo of others.

14. From the fluxions thus determined, the fluxion of any power of a variable quantity (whoſe exponent is a whole poſitive number) is very readily obtained; nothing more being herein required, than to conſider all the factors as equal among themſelves. Thus, the

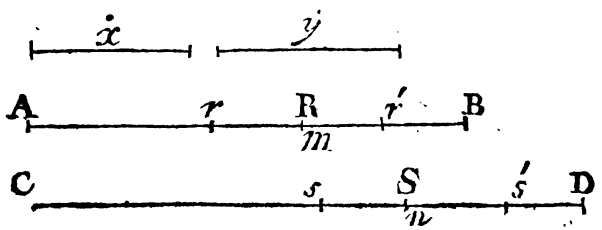
Digitized by Google

the fluxion of $yvw$ being found $\dot{y}vw + y\dot{v}w + yv\dot{w}$, it is plain, if both $v$ and $w$ be ſuppoſed equal to $y$, that the fluent, or quantity propounded, will become $y^3$, and its fluxion $yy\dot{y} + y\dot{y}y + \dot{y}yy = 3y^2\dot{y}$.

Thus, alſo, becauſe the fluxion of $yzvw$ is $yzv\dot{w} + yzv\dot{w} + yz\dot{v}w + y\dot{z}vw$, it appears that the fluxion of $y^4$ will be truly expreſſed by $4y^3\dot{y}$. And, from the given fluxion of $yzvws$, that of $y^5$ will in like manner appear to be $5y^4\dot{y}$. From whence the law of continuation is manifeſt; the fluxion of $y^p$ being univerſally expounded by $py^{p-1}\dot{y}$.

15. This laſt general concluſion, which is of very great importance in the buſineſs of fluxions, being the reſult of ſeveral deductions, whereby its truth and evidence may, perhaps, loſe a part of their force, a more direct inveſtigation of the *ſame* may not here be amiſs; though the former method will, doubtleſs, appear the moſt eaſy and proper for beginners, to whom, the manner of working by general indices, is not plain and familiar.

Conceive two points $m$ and $n$ to move, at the ſame time, from two other points A and C, along the right-lines AB and CD; and let every thing be ſuppoſed as in Prop. 1; *only*, let the meaſure of the diſtance deſcribed by the point $n$ be, *always*, equal to the $p$ power, (inſtead of the ſquare) of that deſcribed by the other point $m$, moving uniformly.




Then,

Digitized by Google

Then, since by hypothesis, the value of CS is here $= x^p$, and that of $Cs = u^p$, *(see the second solution to the foresaid prop.)* it is evident that the distance $s$S, described in the same time with $r$R $(= x-u)$, will be truly defined by $x^p - u^p$, or its equal $x-u \times x^{p-1} + x^{p-2}u + x^{p-3}u^2 + x^{p-4}u^3 + \&c. + u^{p-1}$. Whence, supposing $\dot{x}$ to denote the measure of the uniform velocity of $m$, it will appear, by reasoning as in the said proposition, that the measure of the velocity of $n$, at any place $s$, on this side of S, must be less than $\dot{x} \times x^{p-1} + x^{p-2}u + x^{p-3}u^2 + \&c. + u^{p-1}$, and consequently less than $\dot{x} \times px^{p-1}$; seeing each of the $(p)$ terms of the said series (the first only excepted) is less than $x^{p-1}$, $u$ being less than $x$.

Again, by considering $r'$ and $s'$ as two other, cotemporary positions, beyond R and S, and reasoning in the same manner, the measure of the velocity at $s'$ will appear to be greater, now, than the abovesaid quantity $\dot{x} \times px^{p-1}$.

Therefore, as the velocity increases continually, and seeing the value thereof, before the point arrives at S, is every where less; and afterwards, every where greater, than $\dot{x} \times px^{p-1}$ (or $px^{p-1}\dot{x}$) it is evident, that, at S, it must be neither lesser nor greater, but exactly equal to $px^{p-1}\dot{x}$: which quantity is therefore the true fluxion of $x^p$: and agrees exactly with that determined above.

16. In bringing out the foregoing fluxion of $x^p$, the value of $p$, in both methods, was taken as a whole number: nevertheless the conclusion itself holds equally true, when $p$ is a fraction, as in the notation of roots. To make this appear, let the quantity $x^{\frac{3}{2}}$ $(= \sqrt{x^3})$ be propounded, in order to determine the fluxion thereof: which, by writing $\frac{3}{2}$ for $p$, in the general fluxion

Digitized by Google

fluxion $px^{p-1}\dot{x}$, comes out $\frac{1}{2}x^{\frac{1}{2}}\dot{x}$. Now, to prove that this is the true fluxion, put $y=x^{\frac{1}{2}}$, the quantity given; and then, by ſquaring both ſides, you will have $y^2=x^1$; which, in fluxions, gives $2y\dot{y}=3x^2\dot{x}$ (as has been already ſhewn.) This, by ſubſtituting for $y$, becomes $2x^{\frac{1}{2}}\dot{y}=3x^2\dot{x}$. Whence, by diviſion, $\dot{y}=\frac{3}{2}x^{\frac{1}{2}}\dot{x}$, *the very ſame as before.*

But, to demonſtrate the ſame thing in a general manner, ſuppoſe the fluxion of $x^{\frac{m}{n}}$ to be required ($m$ and $n$ being any whole numbers, whatever) put $y=x^{\frac{m}{n}}$; and then, by raiſing both ſides to the power $n$, you will have $y^n=x^m$: which, in fluxions, gives $ny^{n-1}\dot{y}=mx^{m-1}\dot{x}$; and, conſequently, $\dot{y}=\frac{m}{n}\times\frac{x^{m-1}\dot{x}}{y^{n-1}}=\frac{m}{n}\times\frac{yx^{m-1}\dot{x}}{y^n}=\frac{m}{n}\times\frac{yx^{m-1}\dot{x}}{x^m}=\frac{m}{n}\times\frac{x^{\frac{m}{n}}x^{m-1}\dot{x}}{x^m}=\frac{m}{n}\times x^{\frac{m}{n}-1}\dot{x}$.

Which value, of the fluxion of $x^{\frac{m}{n}}$, is evidently the very ſame, as that ariſing by expounding $p$ in the general fluxion ($px^{p-1}\dot{x}$) by $\frac{m}{n}$; *which was to be proved.*

17. Now, from what has been thus far delivered, the following practical rules, for determining the fluxions of algebraic quantities, are obtained.

## RULE I.

### To find the Fluxion of any given Power of a flowing Quantity.

*Multiply the fluxion of the root by the exponent of the given power, and the product into that power of the ſame root*

Digitized by Google

*root which ariſes by ſubtracting unity from the given exponent.*

The reaſon of this rule is ſeen above; the rule itſelf being nothing more than $p\dot{x}x^{p-1}$ (the fluxion of $x^p$) expreſſed in words.

## RULE II.

To find the Fluxion of the Product of ſeveral variable Quantities, multiplied together.

*Multiply the fluxion of each, by the product of the reſt of the quantities; ſo ſhall the ſum of all the products thus ariſing be the true fluxion required.*

The reaſon of which is, likewiſe, evident from what has been already delivered. *See Art.* 13.

## RULE III.

To find the Fluxion of a Fraction, ariſing from the Diviſion of one variable Quantity by another.

*From the fluxion of the numerator, drawn into the denominator, ſubtract the fluxion of the denominator, drawn into the numerator; and divide the remainder by the ſquare of the denominator.*

This appears from $\frac{y\dot{z}-z\dot{y}}{yy}$, the fluxion of $\frac{z}{y}$, determined in *Art.* 12.

Though I might here, with propriety enough, put an end to this part, as my profeſſed deſign therein extends no farther than giving the young beginner ſome account of the nature, and firſt principles, of fluxions, together with the inveſtigation of the fundamental rules exhibited above; nevertheleſs, as different ways of bringing out the ſame truths have often a very good effect, and ſeeing fluxions of all quantities whatever (whether powers, fractions, *&c.*) are deducible from the fluxion of a rectangle, I ſhall, thefore, ſubjoin a different method, whereby the ſaid fluxion

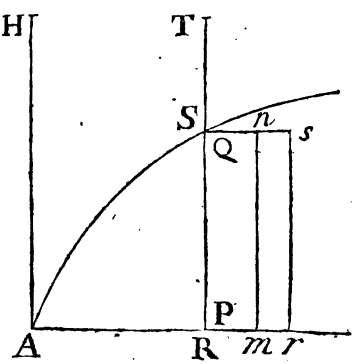
Digitized by Google

fluxion of a rectangle *(given by Prop.* 2.) may be inveſtigated: in order to which it will be requiſite, firſt of all, to premiſe the following

## LEMMA.

18. *The Fluxion of a curvilineal Space* ARS, *generated by the Ordinate* RS *(or the intercepted Part of a Right-line* R r *(moving parallel to itſelf, is equal to the Rectangle* (Rs) *under the ſaid Ordinate, and the Fluxion* (Rr) *of the Abſciſſa* AR.

For, let a right-line *mn*, of the ſame length with RS in the propoſed poſition PQ, be conceived to move from thence, parallel to itſelf, with the ſame celerity that the generating line itſelf has in that poſition: by which means the rectangle P*r*s*s*Q will be *uniformly* generated, with the very celerity by which it begins to be generated, or, by which the ſpace ARS is encreaſed in the propoſed poſition PQ; ſince both the length, and velocity of *mn*, are the ſame as thoſe of RS in the ſaid poſition. Hence the rectangle ſo generated muſt, conſequently, be the true fluxion of the ſpace ARS, *by the definition.*

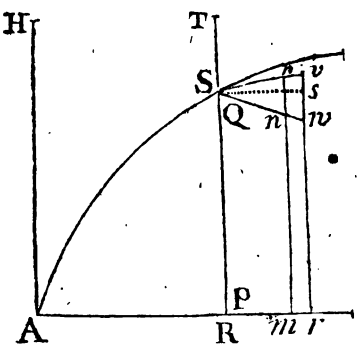



But the ſame thing may be otherwiſe made to appear, from a different, and more logical method of arguing; by proving that the required fluxion can neither be greater, nor leſs, than the ſaid rectangle.



Digitized by Google

Thus, if the line *mn* (while it moves uniformly on towards *rs*) be ſuppoſed to increaſe in length, the area P*mn*Q, generated thereby, will evidently be greater than that which would uniformly ariſe in the ſame time, with the given length at the firſt poſition PQ; ſince the new parts, produced each ſucceeding moment (as the generating line continues to lengthen) are greater and greater.




And, in the ſame manner, if the line *mn*, generating the fluxion, be ſuppoſed to decreaſe, in length, from the given poſition PQ, it is equally plain that the generated ſpace P*mn*Q will be leſs than the cotemporary ſpace that would, uniformly, ariſe with the given length at the firſt poſition PQ.

Therefore, ſeeing the fluxion (or the ſpace that would uniformly ariſe from the generating celerity at the propoſed poſition) is leſs than any ſpace that can be deſcribed, in the given time, when the line *mn* increaſes, and greater than any ſpace that can be deſcribed, when the ſaid line decreaſes; it muſt conſequently be equal to that ſpace which will ariſe, when the length of the ſaid line, from the given poſition, is ſuppoſed neither to increaſe nor decreaſe; that is, when the generated ſpace P*mn*Q is a rectangle, as in the preceding figure.

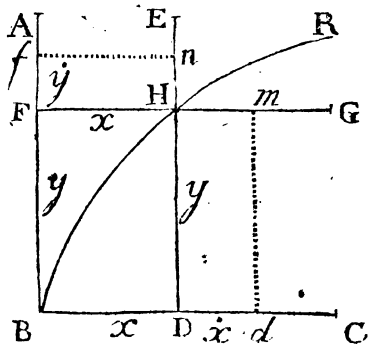
PROPOSITION

Digitized by Google

## PROPOSITION III.

19. *To determine the Fluxion of the Product or Rectangle, of two variable Quantities* (x *and* y.)

Let two right-lines DE and FG, perpendicular to each other, be conceived to move from two other right-lines BA and BC, continually parallel to themſelves, and thereby generate the variable rectangle DF: let the path of their interſection, or the place of the angle H, be the line BHR, dividing the generated rectangle DF in two parts, BDH and BHF: moreover let D*d* $(\dot{x})$ and F*f* $(\dot{y})$ be the fluxions of the ſides BD $(x)$ and BF $(y)$; and ſuppoſe *dm* and *fn* to be drawn parallel, and equal, to DH and FH, reſpectively.




Then, ſince, *by the preceding lemma*, the fluxion of the ſpace or area BDH, is truly expreſſed by the rectangle D*m* $(=x\dot{y})$, and that of the ſpace or area BFH, by the rectangle F*n* $(=y\dot{x})$, it follows, becauſe equal quantities have equal fluxions, that the fluxion of the propoſed rectangle $xy$ (=BDH+BFH) is truly expreſſed by $x\dot{y}+y\dot{x}$, the very expreſſion before determined. *See Art.* 11.

It

It may, perhaps, be expected, that I ſhould now give ſome inſtances of the uſe and application of the theory hitherto explained, in the reſolution of problems: but, having inſiſted very largely on this head in my *Doctrine and Application of Fluxions*, I ſhall take the liberty to recommend that work, to the peruſal of ſuch as are deſirous of farther information in the matter. Thoſe, for whoſe uſe the above account is in a more particular manner deſigned, may have the opportunity of being inſtructed in the practice, by proper examples, without the trouble of turning to other books.

FINIS.

Digitized by Google

Digitized by Google

335 AUC 5

Digitized by Google